

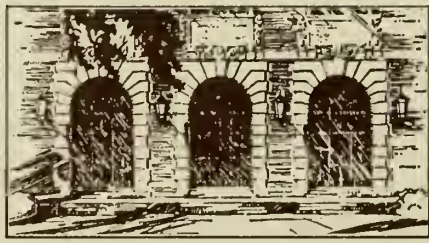
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REPORT NO. 131

NOTE ON A PAPER BY A. NORDSIECK

by

Jean Descloux

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TABLE OF CONTENTS

	Page
INTRODUCTION	1
1. THE ADAMS METHODS	2
2. AN ASYMPTOTIC RELATION FOR THE ERROR OF INTEGRATION WITH THE ADAMS METHODS	4
3. SELF-STABILIZING PROPERTY OF THE ADAMS METHODS	18
4. THE NORDSIECK METHOD	22
5. PREDICTOR FORMULAS	36
6. MODIFICATION OF THE LENGTH OF THE STEP	37
7. STARTING PROCEDURES	42
8. CONCLUSION	56
9. TABLE OF RESULTS	57
APPENDIX 1	70
APPENDIX 2. SOME EQUIVALENT "METHODS"	72

INTRODUCTION

This report is an attempt to analyze Nordsieck's paper [1] with the help of the asymptotic methods developed by Henrici in [2].

The definitions of the Adams-Bashforth formulas and the Adams-Moulton formulas are contained in Section 1.

An asymptotic relation for the errors in the Adams methods is derived in Section 2. Most of the arguments used are directly suggested by [2].

Section 3 is concerned with the self-stabilizing property of the Adams methods.

Section 4 shows how to get Nordsieck-like formulas from multistep methods; applying this to the Adams-Moulton method, one gets the Nordsieck formulas; a modified Nordsieck formula is suggested.

Predictor formulas for the Nordsieck method and the modified Nordsieck formula are examined in Section 5 and interpreted in the language of multistep methods.

The errors involved in the process of changing the step are analyzed in Section 6.

Section 7 deals with starting procedures.

Section 8 gives some conclusions.

A table of results is presented in Section 9.

Appendix 1 is a complement of Section 2.

In Appendix 2, a formal generalization of the Nordsieck method is discussed; it turns out that these formulas are "almost equivalent" to the classical multistep methods. As a particular case, to every multistep method corresponds an equivalent method using less "past values."

No serious practical computations have been done for checking the theoretical results of this report. However, some verifications of the Nordsieck method and of the modified Nordsieck method, $k = 2$, with the differential equation $y' = y$, have been satisfactory.

Note: In this report, all variables are real.

SECTION 1. THE ADAMS METHODS

Let $y(x)$ be a function and $y'(x)$ be its derivative.

Adams-Bashforth Formulas (Predictor Type)

Let $P(x)$ be a polynomial of degree $k + 1$ satisfying the relations

$$P(x_0) = y(x_0)$$

$$P'(x_0 - ih) = y'(x_0 - ih), \quad i = 0, 1, \dots, k;$$

$P(x_0 + h)$ is the result of the k^{th} Adams-Bashforth formula. It is easy to show that there exist the constants (independent of h, y, x) $\beta_0, \beta_1, \dots, \beta_k$ such that

$$P(x_0 + h) = y(x_0) + h(\beta_0 y'(x_0) + \beta_1 y'(x_0 - h) + \dots + \beta_k y'(x_0 - kh));$$

if $y(x)$ possesses $k + 2$ continuous derivatives, the difference $L(y(x_0), h) = y(x_0 + h) - P(x_0 + h)$ satisfies the relation

$$L(y(x_0), h) = D_k h^{k+2} y^{(k+2)}(\xi), \quad x_0 - kh \leq \xi \leq x_0 + h,$$

where D_k is a constant.

Adams-Moulton Formula (Corrector Type)

Let $Q(x)$ be the polynomial of degree $k + 2$ satisfying the relations

$$Q(x_0) = y(x_0)$$

$$Q'(x_0 - ih) = y'(x_0 - ih), \quad i = -1, 0, \dots, k;$$

$Q(x_0 + h)$ is the result of the k^{th} Adams-Moulton formula and satisfies a relation of the form

$$Q(x_0 + h) = y(x_0) + h(\beta_{-1}y'(x_0 + h) + \beta_0y'(x_0 - L) \dots + \beta_ky'(x_0 - kh))$$

where $\beta_{-1}, \beta_0, \dots, \beta_k$ are constant. If $y(x)$ possesses $k + 3$ continuous derivatives, the difference $L(y(x_0), h) = y(x_0 + h) - Q(x_0 + h)$ is given by

$$L(y(x_0), h) = C_k h^{k+3}(\xi), \quad x_0 - kh \leq \xi \leq x_0 + h,$$

where C_k is a constant.

SECTION 2.

AN ASYMPTOTIC RELATION FOR THE ERROR OF INTEGRATION WITH THE ADAMS METHODS

Notations

Any letter with an arrow is a vector of dimension m ; the components of any vector \vec{y} are denoted by y_1, y_2, \dots, y_m . $||\vec{y}|| = \sum_{i=1}^m |y_i|$ is the norm of \vec{y} .

We shall consider the integration of a system of m differential equations on the interval $b \leq x \leq c$; however, in order to allow the possibility of initial values for $x \leq b$, we shall give the assumptions on the differential equations for the interval $a \leq x \leq c$, where $a \leq b$.

h denotes the step of integration and we write $x_n = b + nh$, where n is a positive or negative integer. For any function $y(x)$ defined on the complete interval $[a, c]$ or only at the discrete points x_n , we use the notation $y(x_n) = y_n$.

If t is a variable, $O(t)$ represents a quantity such that $|O(t)| \leq w|t|$, where w is independent of t ; t may be restricted to some interval. $\vec{O}(t)$ is a vector such that $||\vec{O}(t)|| = O(t)$. We shall use in the following notations of this type:

$$y(x) = z(x) + O(h^p);$$

we always suppose that $O(h^p)$ is uniform in x for $a \leq x \leq c$, i.e., there exists a constant w , independent of x and h such that $|O(h^p)| \leq w|h^p|$ for $a \leq x \leq c$.

For any function $y(x)$, $y^{(i)}(x)$ denotes its i^{th} derivative.

Theorem

Let

$$\frac{d}{dx} \vec{y}(x) = \vec{f}(x, \vec{y}) \tag{1}$$

be a system of m differential equations and

$$g_{ij}(x, \vec{y}) = \frac{\partial f_i(x, \vec{y})}{\partial y_j}, \quad i, j = 1, 2, \dots, m$$

be the elements of the matrix $G(x, \vec{y})$.

We assume that for any \vec{y} and \vec{y}^* , $i, j = 1, 2, \dots, m$, $a \leq x \leq c$:

a) $g_{ij}(x, \vec{y})$ exists and is continuous in x and \vec{y} ;

$$b) |g_{ij}(x, \vec{y})| \leq \frac{\Omega}{m}, \quad \Omega \text{ constant} \quad (2)$$

$$c) |g_{ij}(x, \vec{y}^*) - g_{ij}(x, \vec{y})| \leq \psi ||\vec{y}^* - \vec{y}||, \quad \psi \text{ constant} \quad (3)$$

Let $\vec{z}(x)$ be a particular solution of (1); we assume that for $a \leq x \leq c$.

d) $\vec{z}(x)$ possesses $(p + 2)$ continuous derivatives (see (5));

e) $g_{ij}(x, \vec{z}(x))$ has a continuous derivative in x , $i, j = 1, \dots, m$

Let

$$L(y(x), h) = y(x + h) - y(x) - h(\beta_{-1}y'(x + h) + \dots + \beta_k y'(x - kh)) \quad (4)$$

be any formula described in Section 1 for $k \geq 1$; if $\beta_{-1} = 0$, it is an Adams-Bashforth formula; if $\beta_{-1} \neq 0$, it is an Adams-Moulton formula; let C and p ($p \geq 2$) be such that

$$L(y(x), h) = Ch^{p+1} y^{(p+1)}(\xi), \quad x - kh \leq \xi \leq x + h \quad (5)$$

Finally let \vec{u}_n satisfy the difference equation

$$\vec{u}_{n+1} = \vec{u}_n + h(\beta_{-1}\vec{f}_{n+1} + \beta_0\vec{f}_n + \dots + \beta_k\vec{f}_{n-k}) \quad (6)$$

where $\vec{f}_n = \vec{f}(x_n, \vec{u}_n)$ for $n \geq q$; q is a given integer such that $x_{q-k} \geq a$; the initial values $\vec{u}_q, \vec{f}_{q-1}, \dots, \vec{f}_{q-k}$ satisfy the relations

$$f) \vec{u}_q = \vec{z}_q + h^s \vec{\delta} + \vec{O}(h^{s+1}), \quad \vec{\delta} \text{ constant, } s \text{ integer } \geq 1 \quad (7)$$

$$\vec{f}_{q-i} = \vec{z}'_{q-i} + h^r \vec{\eta}_i + \vec{O}(h^{r+1}), \quad i = 1, 2, \dots, k, \quad \vec{\eta}_i \text{ constant, } r \text{ integer } \geq 1. \quad (8)$$

Remark that $\vec{u}_{q-1}, \vec{u}_{q-2}, \dots, \vec{u}_{q-k}$ are not necessarily defined. (6) is an implicit equation for \vec{u}_{n+1} ; in order to provide the existence of a unique solution, we assume (see Appendix 1) that $h \leq \bar{h}$, where

$$g) \quad \bar{h} \leq \frac{1}{2\Omega(|\beta_{-1}| + |\beta_0| + \dots + |\beta_k|)} \quad (9)$$

Then, under hypotheses a to g, the error $\vec{d}(x) = \vec{u}(x) - \vec{z}(x)$ is given by

$$\vec{d}(x) = h^p(\vec{e}_I(x) + \vec{O}(h)) + h^s(\vec{e}_{II}(x) + \vec{O}(h)) + h^{r+1}(\vec{e}_{III}(x) + \vec{O}(h)) \quad (10)$$

where $\vec{e}_I, \vec{e}_{II}, \vec{e}_{III}$ are the solutions of the systems of differential equations

$$\frac{d}{dx} \vec{e}_I(x) = G(x, \vec{z}(x))\vec{e}_I(x) - C\vec{z}^{(p+1)}(x), \quad \vec{e}_I(b) = 0; \quad (11)$$

$$\frac{d}{dx} \vec{e}_{II}(x) = G(x, \vec{z}(x))\vec{e}_{II}(x), \quad \vec{e}_{II}(b) = \vec{\delta}; \quad (12)$$

$$\frac{d}{dx} \vec{e}_{III}(x) = G(x, \vec{z}(x))\vec{e}_{III}(x), \quad \vec{e}_{III}(b) = \vec{\eta}; \quad (13)$$

$$\vec{\eta} = (\beta_1 + \beta_2 + \dots + \beta_k)\vec{\eta}_1 + (\beta_2 + \beta_3 + \dots + \beta_k)\vec{\eta}_2 + \dots + \beta_k\vec{\eta}_k.$$

Generalized Theorem

In the preceding theorem, we have assumed that $\vec{z}(x)$ is a fixed solution of (1). In fact, the same conclusions hold if $\vec{z}(x)$ depend on h , provided we add the two following assumptions:

h) there exists a constant U , independent of x and h such that for $a \leq x \leq c, 0 \leq h \leq \bar{h}$

$$||\vec{z}^{(p+2)}(x)|| \leq U;$$

i) there exists a constant V such that

$$|g_{ij}(x, \vec{z}(x))| \leq V, \quad i, j = 1, \dots, m.$$

Remarks

- 1) The assumptions b and c are very strong since we ask them to be valid for all \vec{y} and \vec{y}^* ; however, it is possible a posteriori to weaken them, since in the proof of the theorem we shall use only a bounded region of \vec{y} .
- 2) The proof of the generalized theorem is almost word for word the same as the proof of the theorem itself and will consequently not be given here.

Proof of the Theorem

All the symbols introduced so far in Section 2 keep their meaning in the following.

Lemma 1.

Under assumptions b and c, for $a \leq x \leq c$ and any \vec{y} and \vec{y}^* , we have:

- a) $||\vec{y}^* + \vec{y}|| \leq ||\vec{y}^*|| + ||\vec{y}||$;
- b) $||\vec{f}(x, \vec{y}^*) - \vec{f}(x, \vec{y})|| \leq \Omega ||\vec{y}^* - \vec{y}||$;
- c) $\vec{f}(x, \vec{y}^*) - \vec{f}(x, \vec{y}) = G(x, \vec{y})(\vec{y}^* - \vec{y}) + \vec{O}(|\vec{y}^* - \vec{y}|^2)$
- d) $||G(x, \vec{y})\vec{y}^*|| \leq \Omega ||\vec{y}^*||$

Proof

Lemma 1a is a direct consequence of the properties of the absolute values. For each component $f_i(x, \vec{y})$ of $\vec{f}(x, \vec{y})$, we have the mean-value theorem

$$f_i(x, \vec{y}^*) - f_i(x, \vec{y}) = \sum_{j=1}^m g_{ij}(x, \vec{y}^{**})(y_j^* - y_j) \quad \text{for some } \vec{y}^{**}; \quad (14)$$

$$|f_i(x, \vec{y}^*) - f_i(x, \vec{y})| \leq \frac{\Omega}{m} \sum_{j=1}^m |y_j^* - y_j| = \frac{\Omega}{m} ||\vec{y}^* - \vec{y}||;$$

$$||\vec{f}(x, \vec{y}^*) - \vec{f}(x, \vec{y})|| \leq \Omega ||\vec{y}^* - \vec{y}||,$$

which proves lemma 1b.

In order to prove lemma lc, we start with equations (14); by (3):

$$g_{ij}(x, \vec{y}^{**}) = g_{ij}(x, \vec{y}) + O(||\vec{y}^{**} - \vec{y}||);$$

since $y_j^* - y_j = O(||\vec{y}^* - \vec{y}||)$ and $||\vec{y}^{**} - \vec{y}|| \leq ||\vec{y}^* - \vec{y}||$, it follows that

$$f_i(x, \vec{y}^*) - f_i(x, \vec{y}) = \sum_{j=1}^m g_{ij}(x, \vec{y})(\vec{y}_j^* - \vec{y}_j) + O(||\vec{y}^* - \vec{y}||^2),$$

which is the desired result for the i^{th} component.

Lemma ld can be proved in the same way as lemma lb.

Lemma 2

Let $y_{-k}, y_{-k+1}, \dots, y_0, y_1, y_2, \dots$ satisfy the inequalities

$$y_{n+1} \leq y_n + h(b_{-1}y_{n+1} + b_0y_n + \dots + b_ky_{n-k}) + \gamma \quad \text{for } n \geq 0; \quad (15)$$

$$0 \leq y_{-i} \leq Y, \quad i = 1, 2, \dots, k; \quad y_0 \geq 0; \quad b_i \geq 0, \quad i = -1, 2, \dots, k; \quad \gamma \geq 0;$$

let $B = b_{-1} + b_0 + \dots + b_k$ and suppose that

$$h > 0, \quad B > 0, \quad h \leq \frac{1}{2B}; \quad (16)$$

then

$$y_n \leq (4^k(y_0 + hBY + \gamma) + \frac{\gamma}{hB})e^{3nhB} \quad \text{for } n \geq 0. \quad (17)$$

Proof

From (16), it follows that

$$1 + hB \leq 1.5;$$

$$\frac{1}{1 - hB} = 1 + \frac{hB}{1 - hB} \leq 1 + 2hB \leq (1 + hB)^2; \quad (18)$$

from (16) and (18), we get

$$y_{n+1} \leq (1 + hB)^2 \{y_n + h(b_0 y_n + b_1 y_{n-1} + \dots + b_k y_{n-k}) + \gamma\};$$

since $y_{-i} \leq Y$ for $i = 1, 2, \dots, k$, we can write for $n = 0, 1, \dots, k-1$:

$$y_{n+1} \leq (1 + hB)^2 \{y_n + (b_0 y_n + b_1 y_{n-1} + \dots + b_n y_0) + hBY + \gamma\};$$

from the two last inequalities, it follows by induction that

$$y_n \leq v_n \text{ for } n \geq 0, \text{ where } v_0 = y_0, \text{ and}$$

$$v_{n+1} = (1 + hB)^2 \{v_n + h(b_0 v_n + b_1 v_{n-1} + \dots + b_n v_0) + hBY + \gamma\} \quad \text{for } 0 \leq n \leq k-1;$$

$$v_{n+1} = (1 + hB)^2 \{v_n + h(b_0 v_n + b_1 v_{n-1} + \dots + b_k v_{n-k}) + \gamma\} \quad \text{for } n \geq k$$

it follows that $v_0 \leq v_1 \leq v_2 \dots$; consequently

$$v_{n+1} \leq (1 + hB)^2 \{(1 + hB)v_n + hBY + \gamma\} \quad \text{for } 0 \leq n \leq k-1;$$

$$v_{n+1} \leq (1 + hB)^2 \{(1 + hB)v_n + \gamma\} \quad \text{for } n \geq k;$$

by induction again, since $(1 + hB)^2 < 3$, $(1 + hB)^3 < 4$, we have

$$v_n \leq w_n \text{ for } n \geq 0, \text{ where } w_0 = y_0 \text{ and}$$

$$w_{n+1} = 4w_n + 3(hBY + \gamma) \quad \text{for } 0 \leq n \leq k-1; \quad (19)$$

$$w_{n+1} = (1 + hB)^3 w_n + 3\gamma \quad \text{for } n \geq k; \quad (20)$$

(19) is a difference equation the solution of which is given (as it is easy to prove by induction) by

$$w_n = 4^n w_0 + 3 \frac{4^n - 1}{4 - 1} (hBY + \gamma) \quad \text{for } 0 \leq n \leq k;$$

in particular, for $n = k$

$$w_k = 4^k w_0 + (4^k - 1)(hBY + \gamma) \leq 4^k (w_0 + hBY + \gamma); \quad (21)$$

in the same way, the solution of (20) is

$$w_n = (1 + hB)^{3(n-k)} w_k + \frac{(1 + hB)^{3(n-k)} - 1}{(1 + hB)^3 - 1} 3\gamma \quad \text{for } n \geq k;$$

clearly, we can write for $n \geq 0$:

$$w_n \leq (1 + hB)^{3n} w_k + 3\gamma \frac{(1 + hB)^{3n} - 1}{(1 + hB)^3 - 1};$$

from the inequalities

$$(1 + hB)^{3n} \leq e^{3nhB}, \quad n \geq 0,$$

$$\frac{1}{(1 + hB)^3 - 1} \leq \frac{1}{3hB},$$

and by (21), we get

$$w_n \leq (4^k (w_0 + hBY + \gamma) + \frac{\gamma}{hB}) e^{3nhB}, \quad n \geq 0$$

which is the desired result.

Lemma 3

If a function $y(x)$ possesses a second continuous derivative on $a \leq x \leq c$, then

$$L(y(x), h) = O(h^2).$$

Proof

First consider the case $y(x) = x$. By (5) it follows that $L(x, h) = 0$, i.e.,

$$\beta_{-1} + \beta_0 + \dots + \beta_k = 1; \quad (22)$$

now, let $y(x)$ be any function satisfying the hypothesis; since $y'(x)$ is absolutely continuous on $a \leq x \leq c$, $y'(x + \theta) = y'(x) + O(h)$ for any $a \leq x - kh \leq \theta \leq x + h \leq c$.

$$L(y(x), h) = y(x) + h(y'(x) + O(h)) - y(x) - hy'(x)(\beta_{-1} + \beta_0 + \dots + \beta_k + O(L));$$

by (22), $L(y(x), h) = O(h^2)$, q.e.d.

Lemma 4

If $\vec{e}(x)$ is the solution of the systems (11) or (12), under assumptions d and e, then $\vec{e}(x)$ possesses a continuous second derivative.

Proof

By lemma 1d, the system of differential equations have a Lipschitz condition which insures, under assumptions d and e a first continuous derivative for $\vec{e}(x)$ (see for example [2]). The lemma 4 results by differentiating the equations (11) or (12).

Lemma 5

Let $\vec{t}(x, \vec{y})$ be a vector satisfying the Lipschitz condition

$$||\vec{t}(x, \vec{y}^*) - \vec{t}(x, \vec{y})|| \leq \Pi ||\vec{y}^* - \vec{y}||, \quad a \leq x \leq c, \Pi \text{ constant.}$$

Suppose that the function $\vec{y}(x)$ satisfies the system

$$\frac{d}{dx} \vec{y}(x) = \vec{t}(x, \vec{y}(x)), \quad a \leq x \leq c,$$

and that \vec{v}_n satisfies the difference equation

$$\vec{v}_{n+1} = \vec{v}_n + h(\beta_{-1}\vec{t}_{n+1} + \beta_0\vec{t}_n + \dots + \beta_k\vec{t}_{n-k}) + \vec{\gamma}_n, \quad (23)$$

where $\vec{t}_n = \vec{t}(x_n, \vec{v}_n)$ for $n \geq j$; j is a given fixed integer (positive or negative) such that $x_{j-k} \geq a$; let $\vec{w}_n = \vec{v}_n - \vec{y}_n$ and $\vec{L}(\vec{y}(x), h)$ be the vector of components $L(y_i(x), h)$. If

$$1) \quad ||\vec{w}_n|| \leq W, \quad n = j-1, j-2, \dots, j-k, \quad W \text{ constant}$$

$$2) \quad ||\vec{\gamma}_n|| \leq \Gamma, \quad n \geq j, \quad \Gamma \text{ constant}$$

$$3) \quad ||\vec{L}(\vec{y}(x), h)|| \leq \Lambda, \quad a \leq x \leq c, \quad \Lambda \text{ constant}$$

then

$$||\vec{w}(x)|| \leq (4^k(||\vec{w}_j|| + hBW + \Gamma + \Lambda) + \frac{\Gamma + \Lambda}{hB})e^{3(c-a)B} \quad \text{for } x_j \leq x \leq c,$$

where $B = \Pi(|\beta_{-1}| + |\beta_0| + \dots + |\beta_k|)$ and $h \leq \frac{1}{2B}$.

Proof

If we subtract the relation

$$\vec{y}_{n+1} = \vec{y}_n + h(\beta_{-1}\vec{y}'_{n+1} + B_0\vec{y}'_n + \dots + \beta_k\vec{y}'_{n-k}) + \vec{L}(\vec{y}_n, h)$$

from (23), we get

$$\vec{w}_{n+1} = \vec{w}_n + h(\beta_{-1}(\vec{t}_{n+1} - \vec{y}'_{n+1}) + \dots + \beta_k(\vec{t}_{n-k} - \vec{y}'_{n-k}) + \vec{\gamma}_n - \vec{L}(\vec{y}_n, h));$$

Since $||\vec{t}_n - \vec{y}'_n|| \leq \Pi||\vec{w}_n||$, we have by lemma 1a:

$$||\vec{w}_{n+1}|| \leq ||\vec{w}_n|| + h(\Pi|B_{-1}| ||\vec{w}_{n+1}|| + \dots + \Pi|\beta_k| ||\vec{w}_{n-k}||) + \Gamma + \Lambda;$$

by lemma 2, if we set $y_n = ||\vec{w}_{n+j}||$, $\gamma = \Gamma + \Lambda$, $Y = W$, $B = \Pi(|\beta_{-1}| + \dots |\beta_k|)$, we get for $n \geq 0$

$$||\vec{w}_{n+j}|| \leq (4^k(||\vec{w}_j|| + hBW + \Gamma + \Lambda) + \frac{\Gamma + \Lambda}{hB})e^{3nhB};$$

since $a < x_j \leq x_{n+j} \leq c$, it follows that $nh \leq c - a$, q.e.d.

Proof of the Theorem

We first consider a particular case. Suppose that for $i = 1, 2, \dots, k$

- 1) \vec{u}_{q-i} are defined;
- 2) $\vec{f}_{q-i} = \vec{f}(x_{q-i}, \vec{u}_{q-i})$;
- 3) $\vec{u}_{q-i} = \vec{z}_{q-i} + \vec{O}(h^s)$; (see 7).

From the third condition, because of lemma 1b, we have

$$\vec{f}_{q-i} = \vec{z}'_{q-i} + \vec{O}(h^s).$$

Since by lemma 1b, $\vec{f}(x, \vec{y})$ satisfies a Lipschitz condition with constant Ω , we can apply lemma 5, replacing \vec{t} by \vec{f} , Π by Ω , $\vec{y}(x)$ by $\vec{z}(x)$, \vec{v}_n by \vec{u}_n , \vec{w}_n by \vec{d}_n , j by q , Γ by 0 , W and $||\vec{w}_j||$ by $O(h^s)$, Λ by $O(h^{p+1})$ (see (5)); we get

$$||\vec{d}(x)|| \leq (4^k(O(h^s) + hB(O(h^s) + O(h^{p+1}))) + \frac{O(h^{p+1})}{hB})e^{3(c-a)B} = O(h^s) + O(h^p). \quad (24)$$

Now, let us subtract from (6) the corresponding equation for $z(x)$:

$$\vec{d}_{n+1} = \vec{d}_n + h(\beta_{-1}(\vec{f}_{n+1} - \vec{z}'_{n+1}) + \beta_0(\vec{f}_n - \vec{z}'_n) + \dots + \beta_k(\vec{f}_{n-k} - \vec{z}'_{n-k})) - \vec{L}(\vec{z}_n, h);$$

by lemma 1c, for $n \geq q - k$:

$$\vec{f}_n - \vec{z}'_n = \vec{f}(x, \vec{u}_n) - \vec{f}(x, \vec{z}_n) = G(x_n, \vec{z}_n)\vec{d}_n + \vec{O}(||\vec{d}_n||^2).$$

We introduce the notations

$$G(x) = G(x, \vec{z}(x)); \quad G_n = G(x_n).$$

By (24) we can write for $n \geq q$

$$\vec{d}_{n+1} = \vec{d}_n + h(\beta_{-1} G_{n+1} \vec{d}_{n+1} + \beta_0 G_n \vec{d}_n + \dots + \beta_k G_{n-k} \vec{d}_{n-k}) - \vec{L}(\vec{z}_n, h) + \vec{O}(h^{2p+1}) + \vec{O}(h^{2s+1});$$

on another side by assumption d, (5) and (22), we have

$$\vec{L}(\vec{z}_n, h) = Ch^{p+1}(\beta_{-1} \vec{z}_{n+1}^{(p+1)} + \dots + \beta_n \vec{z}_{n-k}^{(p+1)}) + \vec{O}(h^{p+2});$$

then

$$\begin{aligned} \vec{d}_{n+1} = \vec{d}_n + h(\beta_{-1}(G_{n+1} \vec{d}_{n+1} - Ch^p \vec{z}_{n+1}^{(p+1)}) + \dots + \beta_k(G_{n-k} \vec{d}_{n-k} - Ch^p \vec{z}_{n-k}^{(p+1)})) \\ + \vec{O}(h^{p+2}) + \vec{O}(h^{2p+1}) + \vec{O}(h^{2s+1}). \end{aligned}$$

We have to distinguish three cases:

First case: $p > s$. Let $\vec{d}_n^* = h^{-s} \vec{d}_n$; \vec{d}_n^* satisfies the difference equation

$$\vec{d}_{n+1}^* = \vec{d}_n^* + h(\beta_{-1} G_{n+1} \vec{d}_n^* + \dots + \beta_k G_{n-k} \vec{d}_{n-k}^*) + \vec{O}(h^2),$$

with the initial conditions

$$\vec{d}_q^* = \vec{\delta} + \vec{O}(h), \quad \vec{d}_{q-i}^* = \vec{O}(1), \quad i = 1, 2, \dots, k;$$

we compare $\vec{d}^*(x)$ with the solution $\vec{e}(x)$ of the system of differential equations

$$\frac{d}{dx} \vec{e}(x) = G(x) \vec{e}(x), \quad \vec{e}(b) = \vec{\delta};$$

Let $\vec{w}_n = \vec{d}_n^* - \vec{e}_n$; we have

$$\vec{w}_q = \vec{O}(h), \quad \vec{w}_{q-i} = \vec{O}(1), \quad i = 1, 2, \dots, k;$$

by lemmas 3 and 4, $\vec{L}(\vec{e}(x), h) = \vec{O}(h^2)$; we can apply lemma 5 with $\vec{t}(x, \vec{y}) = G(x)\vec{y}$, $\Pi = \Omega$ (lemma 1d), $\vec{v}_n = \vec{d}_n^*$, $\vec{y}_n = \vec{e}_n$, $W = O(1)$, $\Gamma = O(h^2)$, $\Lambda = O(h^2)$, $j = q$, $||\vec{w}_q|| = O(h)$; one gets

$$||\vec{w}(x)|| \leq (4^k(O(h) + hBO(1) + O(h^2) + O(h^2) + \frac{O(h^2) + O(h^2)}{hB})e^{3(c-a)B} = O(h),$$

i.e., $\vec{d}^*(x) = \vec{e}(x) + \vec{O}(h)$ and $\vec{d}(x) = h^s(\vec{e}(x) + \vec{O}(h))$, $x_q \leq x \leq c$.

Second case: $p = s$. Let $\vec{d}_n^* = h^{-p}\vec{d}_n$; \vec{d}_n^* satisfies the difference equation

$$\vec{d}_{n+1}^* = \vec{d}_n^* + h(\beta_{-1}(G_{n+1}\vec{d}_n^* - C\vec{z}_{n+1}^{(p+1)}) + \dots + \beta_k(G_{n-k}\vec{d}_{n-k}^* - C\vec{z}_{n-k}^{(p+1)})) + \vec{O}(h^2);$$

using the same arguments as in the first case, we get

$$\vec{d}(x) = h^p(\vec{e}(x) + \vec{O}(h)), \quad x_q \leq x \leq c;$$

where $\vec{e}(x)$ satisfies the differential equation

$$\frac{d}{dx} \vec{e}(x) = G(x)\vec{e}(x) - C\vec{z}^{(p+1)}(x); \quad \vec{e}(b) = \vec{\delta}.$$

Third case: $p < s$. This case is identical to the second one, except that we have to replace $\vec{\delta}$ by 0. Clearly, the following relation is valid for the three cases

$$\vec{d}(x) = h^p(\vec{e}_I(x) + \vec{O}(h)) + h^s(\vec{e}_{II}(x) + \vec{O}(h)),$$

where $\vec{e}_I(x)$ and $\vec{e}_{II}(x)$ satisfy respectively (11) and (12); this finishes the proof of the theorem for the particular case.

Now, it is easy to prove the theorem for the general case. Subtracting from (6) the corresponding equation for $\vec{z}(x)$ for $n = q$ we get

$$\vec{d}_{q+1} = \vec{d}_q + h(\beta_{-1}(\vec{f}_{q+1} - \vec{z}'_{q+1}) + \beta_0(\vec{f}_q - \vec{z}'_q) + \dots + \beta_k(\vec{f}_{q-k} - \vec{z}'_{q-k})) - \vec{L}(\vec{z}_q, h); \quad (25)$$

by lemma 1b, $||\vec{f}_{q+1} - \vec{z}'_{q+1}|| \leq \Omega ||\vec{d}_{q+1}||$; by lemma 1a and equations (5), (7), (8), we have

$$||\vec{d}_{q+1}|| \leq o(h^s) + h(|\beta_{-1}| \Omega ||\vec{d}_{q+1}|| + o(h^s) + o(h^2)) + o(h^{p+1});$$

by (9), $1 - h|\beta_{-1}| \Omega \geq .5$; consequently

$$||\vec{d}_{q+1}|| = o(h^s) + o(h^{r+1}) + o(h^{p+1});$$

using this relation in (25) we get by (7) and (8):

$$\vec{d}_{q+1} = h^s \vec{\delta} + \beta_1 h^{r+1} \vec{\eta}_1 + \beta_2 h^{r+1} \vec{\eta}_2 + \dots + \beta_k h^{r+1} \vec{\eta}_k + \vec{O}(h^{s+1}) + \vec{O}(h^{r+2}) + \vec{O}(h^{p+1}).$$

In the same way, we can obtain for \vec{d}_{q+2} :

$$\begin{aligned} \vec{d}_{q+2} &= \vec{d}_{q+1} + \beta_2 h^{r+1} \vec{\eta}_1 + \beta_3 h^{r+1} \vec{\eta}_2 + \dots + \beta_k h^{r+1} \vec{\eta}_{n-1} + \vec{O}(h^{s+1}) + \vec{O}(h^{r+2}) + \vec{O}(h^{p+1}) \\ &= h^s \vec{\delta} + h^{r+1} (\beta_1 + \beta_2) \vec{\eta}_1 + \dots + h^{r+1} (\beta_{k-1} + \beta_k) \vec{\eta}_{k-1} + h^{r+1} \beta_k \vec{\eta}_k + \vec{O}(h^{s+1}) + \vec{O}(h^{r+2}) + \vec{O}(h^{p+1}); \end{aligned}$$

repeating the same process for $\vec{d}_{q+3}, \dots, \vec{d}_{q+k}$, we finally get

$$\begin{aligned} \vec{d}_{q+k} &= h^s \vec{\delta} + h^{r+1} (\beta_1 + \beta_2 + \dots + \beta_k) \vec{\eta}_1 + h^{r+1} (\beta_2 + \dots + \beta_k) \vec{\eta}_2 + \dots + h^{r+1} \beta_k \vec{\eta}_k \\ &\quad + \vec{O}(h^{s+1}) + \vec{O}(h^{r+2}) + \vec{O}(h^{p+1}) \\ &= h^s \vec{\delta} + h^{r+1} \vec{\eta} + \vec{O}(h^{s+1}) + \vec{O}(h^{r+1}) + \vec{O}(h^{p+1}), \end{aligned}$$

$$\vec{d}_{q+k-i} = \vec{O}(h^s) + \vec{O}(h^{r+1}) + \vec{O}(h^{p+1}), \quad i = 1, 2, \dots, k.$$

We are now in position to apply the theorem proved above for the particular case. It suffices to replace q by $q+k$ and to remark that the system of differential equations

$$\frac{d}{dx} \vec{e}_{II}(x) = G(x) \vec{e}_{II}(x)$$

is linear.

SECTION 3. SELF-STABILIZING PROPERTY OF THE ADAMS METHODS

We use the same notations and conventions as in Section 2. Let $\vec{y} = f(x, \vec{y})$ be a system of m differential equations satisfying the Lipschitz condition

$$||\vec{f}(x, \vec{y}^*) - \vec{f}(x, \vec{y})|| \leq \Omega ||\vec{y}^* - \vec{y}|| \quad \text{for } a \leq x \leq c. \quad (1)$$

Let $\vec{z}(x)$ be a particular solution of the system and \vec{u}_n the approximation obtained by using the k^{th} Adams-Moulton formula (see Section 1):

$$\vec{u}_{n+1} = \vec{u}_n + h(\beta_{-1}\vec{f}_{n+1} + \dots + \beta_k\vec{f}_{n-k}), \quad (2)$$

where $x_n = b + nh$, $\vec{f}_n = \vec{f}(x_n, \vec{u}_n)$ for $n \geq 0$, $0 < h \leq \frac{1}{2\Omega(|\beta_{-1}| + \dots + |\beta_k|)} = \bar{h}$. We suppose that the initial conditions for \vec{u}_n in b are such that

$$\vec{u}(x) = \vec{z}(x) + \vec{O}(h), \quad b \leq x \leq c; \quad (2a)$$

let $\vec{v}(x)$ be the exact solution of the differential equations such that $\vec{v}(c) = \vec{u}(c)$; $\vec{v}(x)$ depends on h ; we suppose that $\vec{v}(x)$ possesses $k+3$ continuous derivatives and that there exists a constant W such that

$$||\vec{v}^{(k+3)}(x)|| \leq W \quad \text{for } b \leq x \leq c, \quad h \leq \bar{h}.$$

Theorem

For any fixed integer r independent of h

$$\vec{u}(c - ih) - \vec{v}(c - ih) = \vec{O}(h^{k+3}), \quad i = 0, 1, \dots, r \quad (3)$$

Remark

The self-stabilizing property allows us to apply the so-called "Milne's method for evaluating the local discretization error" even if the initial conditions are "bad."

Let $x_t = c$ (t depends on L) and \vec{u}_t^* be the value of the predictor obtained by the $(k+1)^{\text{th}}$ Adams-Bashforth formula

$$\vec{u}_t^* = \vec{u}_{t-1} + h(\beta_0^* \vec{f}_{t-1} + \beta_1^* \vec{f}_{t-2} + \dots + \beta_{k+1}^* \vec{f}_{t-k-2}); \quad (4)$$

in Section 1, we have mentioned that

$$\vec{v}_t = \vec{v}_{t-1} + h(\beta_0^* \vec{v}'_{t-1} + \dots + \beta_{k+1}^* \vec{v}'_{t-k-2}) + D_{k+1} h^{k+3} \vec{v}^{(k+3)}(\xi) \quad (5)$$

Subtracting (5) from (4), we get by (1) and (3):

$$\vec{u}_t^* - \vec{v}_t = \vec{u}_{t-1} - \vec{v}_{t-1} - D_{k+1} h^{k+3} \vec{v}^{(k+3)}(\xi) + \vec{O}(h^{k+4}); \quad (6)$$

in the same way, comparing (2) with the analogous equation for $\vec{v}(x)$:

$$\vec{u}_t - \vec{v}_t = \vec{u}_{t-1} - \vec{v}_{t-1} - C_k h^{k+3} \vec{v}^{(k+3)}(\eta) + \vec{O}(h^{k+4}) \quad (7)$$

where $x_{t-k-1} \leq \xi, \eta \leq x_t$.

Subtracting (7) from (6):

$$\vec{u}_t^* - \vec{v}_t = h^{(k+3)} (C_k \vec{v}^{(k+3)}(\eta) - D_{k+1} \vec{v}^{(k+3)}(\xi)) + \vec{O}(h^{k+4}); \quad (8)$$

if we suppose that $\vec{v}^{(k+4)}(x)$ is continuous in x and bounded on $b \leq x \leq c$, $0 < h \leq \bar{h}$, then

$$C_k \vec{v}^{(k+3)}(\eta) - D_{k+1} \vec{v}^{(k+3)}(\xi) = (C_k - D_{k+1}) \vec{v}^{(k+3)}(\eta) + \vec{O}(h);$$

replacing in (8), one gets an approximative value for $\vec{v}^{(k+3)}(\eta)$ which provides an asymptotical evaluation of the truncation error $C_k h^{k+3} \vec{v}^{(k+3)}(\eta)$ in (7).

Proof of the Theorem

Lemma

Under hypotheses (1) and (2a),

$$\vec{u}(x) - \vec{v}(x) = \vec{O}(h), \quad b \leq x \leq c.$$

Proof

By (2a), it suffices to show that $\vec{z}(x) - \vec{v}(x) = \vec{O}(h)$. By (1), we can write

$$\vec{z}(x) - \vec{v}(x) = \vec{z}(c) - \vec{v}(c) + \int_c^x (\vec{f}(\vec{z}(u), u) - \vec{f}(\vec{v}(u), u)) du;$$

$$\begin{aligned} ||\vec{z}(x) - \vec{v}(x)|| &\leq ||\vec{z}(c) - \vec{v}(c)|| + \int_c^x ||\vec{f}(\vec{z}(u), u) - \vec{f}(\vec{v}(u), u)|| du \\ &\leq ||\vec{z}(c) - \vec{v}(c)|| + \Omega \int_x^c ||\vec{z}(u) - \vec{v}(u)|| du; \end{aligned}$$

from this last inequality, it is known that

$$||\vec{z}(x) - \vec{v}(x)|| \leq ||\vec{z}(c) - \vec{v}(c)|| e^{\Omega(c-x)}.$$

Since $\vec{z}(c) - \vec{v}(c) = \vec{z}(c) - \vec{u}(c) = \vec{O}(h)$, the conclusion follows.

Now we go to the proof of the theorem itself. Subtracting the two equations

$$\vec{u}_t = \vec{u}_{t-1} + h(\beta_{-1}\vec{f}_t + \dots + \beta_k f_{t-k-1}), \quad (9)$$

$$\vec{v}_t = v_{t-1} + h(\beta_{-1}\vec{v}'_t + \dots + \beta_k \vec{v}'_{t-k-1}) + c_k h^{k+3} \vec{v}^{(k+3)}(\xi), \quad (10)$$

where $x_t = c$, $x_{t-k-1} \leq \xi \leq x_t$, we get by the lemma and (1)

$$0 = \vec{u}_{t-1} - \vec{v}_{t-1} + \vec{O}(h^2) + \vec{O}(h^{k+3})$$

i.e.,

$$\vec{u}_{t-1} = \vec{v}_{t-1} = \vec{O}(h^2) + \vec{O}(h^{k+3}). \quad (11)$$

Changing t in $t-1$ in (9) and (10), one gets because of (11)

$$\vec{u}_{t-2} - \vec{v}_{t-2} = \vec{O}(h^2) + \vec{O}(h^{k+3});$$

by induction, it follows that $\vec{u}_{t-i} - \vec{v}_{t-i} = \vec{O}(h^2) + \vec{O}(h^{k+3})$ for $i = 0, 1, \dots, s$, where s is any fixed number independent of h ; with this result in mind, we subtract again (10) from (9) and get this time:

$$0 = \vec{u}_{t-1} - \vec{v}_{t-1} + \vec{O}(h^3) + \vec{O}(h^{k+3});$$

by induction it follows that

$$\vec{u}_{t-i} - \vec{v}_{t-i} = \vec{O}(h^3) + \vec{O}(h^{k+3}) \quad \text{for } i = 0, 1, 2, \dots, s-k-1$$

Again by induction for $j = 2, 3, 4, \dots$

$$\vec{u}_{t-i} - \vec{v}_{t-i} = \vec{O}(h^j) + \vec{O}(h^{k+3}) \quad \text{for } i = 0, 1, 2, \dots, s-(j-2)(k+1);$$

for $j = k+3$, we get (3).

SECTION 4. THE NORDSIECK METHOD

Note

In order to simplify the notations and avoid confusion with different kinds of vectors, we shall consider in this section and in the following ones only scalar differential equations; generalizations for systems are always easy and in most of the cases obvious.

Let $y' = f(x, y)$ be a differential equation to be integrated for $x \geq b$ with the help of the general multistep method

$$y_{n+1} = \alpha_0 y_n + \alpha_1 y_{n-1} + \dots + \alpha_j y_{n-j} + h(\beta_{-1} f_{n+1} + \beta_0 f_n + \dots + \beta_k f_{n-k}), \quad n \geq 0 \quad (1)$$

where $\alpha_i, \beta_i, k, j, (j \geq 0)$ are the constants which characterize the formula; some of the α_i 's or β_i 's may vanish; furthermore

$$x_n = b + nh; \quad f_n = f(x_n, y_n) \quad \text{for } n \geq 1; \quad (2)$$

$y_0, y_1, \dots, y_{-j}, f_0, f_{-1}, \dots, f_{-k}$ are the initial values; f_0, f_{-1}, \dots may be different from $f(x_0, y_0), f(x_{-1}, y_{-1}), \dots$

We operate some formal transformation on (1). Let \vec{u}_n be the column vector of elements $y_n, y_{n-1}, \dots, y_{n-j}, f_n, f_{n-1}, \dots, f_{n-k}$ and \vec{p} the constant vector of elements $\alpha_0, \alpha_1, \dots, \alpha_j, h\beta_0, h\beta_1, \dots, h\beta_k$. (1) can be written in the form (\vec{p}^T being the transposed of \vec{p}):

$$y_{n+1} = \vec{p}^T \vec{u}_n + h\beta_{-1} f_{n+1} \quad (3)$$

$$\vec{u}_{n+1} = A \vec{u}_n + f_{n+1} \vec{a} \quad (4)$$

$$f_{n+1} = f(x_{n+1}, y_{n+1}) \quad (5)$$

where

[illegible]

the matrix has been subdivided into four submatrices; the upper-left submatrix is square of order $j+1$; the lower-right submatrix is square of order $k+1$.

For given \vec{u}_0 , (3), (4), (5) define an iterative process; for $n = 0$, (3) and (5) allow to compute f_1 and (4) \vec{u}_1 , etc....

Let T be a regular matrix of order $k+j+2$, the elements of which depend only on h ; we consider the linear transformation

$$\vec{u}_n = T \vec{v}_n;$$

replacing in (3), (4), (5), we get for \vec{v}_n the iterative process

$$y_{n+1} = \overset{\rightarrow T \rightarrow}{q} v_n + h\beta_{-1} f_{n+1} \quad (6)$$

$$\left. \begin{aligned} \vec{v}_{n+1} &= B\vec{v}_n + f_{n+1}\vec{b} \end{aligned} \right\} \text{ for } n \geq 0 \quad (7)$$

$$f_{n+1} = f(x_{n+1}, y_{n+1}) \quad (8)$$

where $\vec{q} = T^T \vec{p}$, $B = T^{-1} A T$, $\vec{b} = T^{-1} \vec{a}$.

The process (6), (7), (8) is equivalent to the process (3), (4), (5) in the following sense: if \vec{u}_0 and \vec{v}_0 are chosen such that $\vec{u}_0 = T\vec{v}_0$, then both processes provide the same values for y_n and furthermore $\vec{u}_n = T\vec{v}_n$ for all $n \geq 0$.

In order to get a particular matrix T , let us consider the polynomial $P_n(x)$ of degree $k+j+1$ satisfying the relations

$$\left. \begin{aligned} P_n(x_{n-i}) &= y_{n-i} & \text{for } i = 0, 1, \dots, j; \\ P'_n(x_{n-i}) &= f_{n-i} & \text{for } i = 0, 1, \dots, k; \end{aligned} \right\} \quad (9)$$

this polynomial exists and is uniquely defined; in order to prove that, it suffices to show that the only polynomial $Q_n(x)$ of degree $k+j+1$ satisfying the conditions

$$Q_n(x_{n-i}) = 0, \quad i = 0, 1, \dots, j$$

$$Q'_n(x_{n-i}) = 0, \quad i = 0, 1, \dots, k$$

is the null polynomial; by Rolle's theorem, there exist $\xi_1, \xi_2, \dots, \xi_j$ such that

$$x_n > \xi_1 > x_{n-1} > \xi_2 > \dots > \xi_j > x_{n-j},$$

$$Q'_n(\xi_i) = 0, \quad i = 1, \dots, j;$$

$Q'_n(x)$ is of degree $k+j$ and possesses at least $k+j+1$ zeros; consequently $Q'_n(x) \equiv 0$ and, since $Q_n(x_n) = 0$, $Q_n(x) \equiv 0$, q.e.d.

Following Nordsieck's notations, let us write $P_n(x)$ in the form

$$P_n(x) = v_n^{(1)} + h(v_n^{(2)} \left(\frac{x - x_n}{h}\right) + v_n^{(3)} \left(\frac{x - x_n}{h}\right)^2 + \dots + v_n^{(k+j+2)} \left(\frac{x - x_n}{h}\right)^{k+j+1}); \quad (10)$$

let \vec{v}_n denote the vector of components $v_n^{(1)}, \dots, v_n^{(k+j+2)}$ and recall that \vec{u}_n has the components $y_n, \dots, y_{n-j}, f_n, \dots, f_{n-k}$; then the system (9) can be written in the form

$$T\vec{v}_n = \vec{u}_n;$$

since the expressions $x_{n-1} - x_n$ depend only on h , so does T ; T is regular, since the system $T\vec{v} = 0$ has only the trivial solution.

The particular choice of T we have got with the help of the polynomial $P_n(x)$ gives to the iterative process (6), (7), (8) two advantageous features:

- 1) Predictor formula. Because of (9), $P_n(x)$ can be considered as a polynomial of approximation of the true solution in the neighborhood of x_n ; consequently $P_n(x_{n+1})$ provides an approximate value for y_{n+1} ; i.e., a predictor value:

$$P_n(x_{n+1}) = v_n^{(1)} + h(v_n^{(2)} + v_n^{(3)} + \dots + v_n^{(k+j+1)});$$

- 2) Modification of the length of the step. Suppose we use the step h for $x \leq x_n$ and the step h^* for $x > x_n$; all the quantities related to h^* will be denoted with a star: $x_n^* = x_n$, $x_{n+1}^* = x_n + h^*$, ...; \vec{q} , B , \vec{b} also depend on h and will be replaced by \vec{q}^* , B^* , \vec{b}^* ; it is natural to choose \vec{v}_n^* such that $P_n^*(x) = P_n(x)$, where

$$P_n^* = v_n^{(1)*} + h^*(v_n^{(2)*} \left(\frac{x - x_n}{h^*} \right) + v_n^{(3)*} \left(\frac{x - x_n}{h^*} \right)^2 + \dots);$$

by comparing with (10), one gets the simple rule:

$$v_n^{(1)*} = v_n^{(1)},$$

$$v_n^{(2)*} = v_n^{(2)},$$

$$v_n^{(3)*} = \frac{h^*}{h} v_n^{(3)},$$

$$v_n^{(k+j+2)*} = \left(\frac{h^*}{h} \right)^{k+j} v_n^{(k+j+2)};$$

by (9) it is easy to interpret the meaning of this rule in terms of the original method (1); denoting by y_n^* , y_{n-1}^* , ..., y_{n-j}^* , f_n^* , ..., f_{n-k}^* the initial values necessary to the integration with step h^* from x_n , we get

$$y_{n-i}^* = P_n(x_n - ih^*) \quad i = 0, 1, \dots, j,$$

$$f_{n-i}^* = P'_n(x_n - ih^*) \quad i = 0, 1, \dots, k;$$

one can observe that generally $f_{n-i}^* \neq f(x_n - ih^*, y_{n-i}^*)$.

The Nordsieck Formulas

If (1) is the Adams-Moulton method described in Section 1 to which one applies the transformation described above, then (6), (7), (8) represent the Nordsieck formula. The formula corresponding to the k^{th} Adams-Moulton method will be called " k^{th} Nordsieck formula;" the corresponding matrix B in (7) is of order $k+2$ and the corresponding polynomial $P_n(x)$ which is of degree $k+1$ is defined by the relations

$$P_n(x_n) = y_n$$

$$P'_n(x_{n-1}) = f_{n-i}, \quad i = 0, 1, \dots, k$$

The Modified Nordsieck Formulas

We can write the k^{th} Adams-Moulton formula in the following way:

$$y_{n+1} = y_n + h\beta_0 f_n + h\beta_{-1} f_{n-1} \dots + h\beta_k f_{n-k} + O f_{n-k-1} + \beta_{-1} f_{n+1};$$

we simply add a zero term; we can apply the same transformation on this formula as we did for the Nordsieck formula; the result will be called the " k^{th} modified Nordsieck formula;" the corresponding matrix B in (7) is of order $k+3$ and $P_n(x)$ which is of order $k+2$ is determined by the relations

$$P_n(x_n) = y_n$$

$$P'_n(x_{n-i}) = f_{n-i}, \quad i = 0, 1, \dots, k+1.$$

Remark

The k^{th} Nordsieck method and the k^{th} modified Nordsieck method are both equivalent to the k^{th} Adams-Moulton method in the following sense: denoting by $P_n(x)$ the polynomial corresponding to the Nordsieck method and $P_n^*(x)$ the polynomial corresponding to the modified Nordsieck method, then all three methods will give the same values for y_n , $n \geq 0$ if

$$P_0(x_0) = P_0^*(x_0) = y_0,$$

$$P'_0(x_{-i}) = P_0'^*(x_{-i}) = f_{-i}, \quad i = 0, 1, \dots, k;$$

the value of $P_0'^*(x_{-k-1})$ has no influence on y_n , $n \geq 0$.

Example

For $k = 1$, the Adams-Moulton method is given by

$$y_{n+1} = y_n + \frac{8h}{12} f_n - \frac{h}{12} f_{n-1} + \frac{5h}{12} f_{n+1};$$

in order to get the Nordsieck formula, we have the sequence of intermediate stages:

$$\vec{p}^T = (1, \frac{8h}{12}, -\frac{h}{12}), \quad \beta_{-1} = \frac{5h}{12};$$

$$\vec{u}_n^T = (y_n, f_n, f_{n-1});$$

$$A = \begin{pmatrix} 1 & \frac{8h}{12} & -\frac{h}{12} \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} \frac{5h}{12} \\ 1 \\ 0 \end{pmatrix};$$

$$P_n(x) = v_n^{(1)} + h \left\{ v_n^{(2)} \left(\frac{x - x_n}{h} \right) + v_n^{(3)} \left(\frac{x - x_n}{h} \right)^2 \right\},$$

$$\vec{v}_n^T = (v_n^{(1)}, v_n^{(2)}, v_n^{(3)});$$

the relations $P_n(x_n) = y_n$, $P'_n(x_n) = f_n$, $P'_n(x_{n-1}) = f_{n-1}$ lead to the relations

$$\vec{u}_n = T \vec{v}_n \quad (11)$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\vec{q}^T = \vec{p}^T T = (1, \frac{7h}{12}, \frac{h}{6})$$

$$B = T^{-1} A T = \begin{pmatrix} 1 & \frac{7h}{12} & \frac{h}{6} \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \end{pmatrix}, \quad \vec{b} = T^{-1} \vec{a} = \begin{pmatrix} \frac{5h}{12} \\ 1 \\ \frac{1}{2} \end{pmatrix};$$

(6) and (7) become explicitly

$$y_{n+1} = v_n^{(1)} + \frac{7h}{12} v_n^{(2)} + \frac{h}{6} v_n^{(3)} + \frac{5h}{12} f_{n+1},$$

$$v_{n+1}^{(1)} = v_n^{(1)} + \frac{7h}{12} v_n^{(2)} + \frac{h}{6} v_n^{(3)} + \frac{5h}{12} f_{n+1};$$

$$v_{n+1}^{(2)} = f_{n+1},$$

$$v_{n+1}^{(3)} = -\frac{1}{2} v_n^{(2)} + \frac{1}{2} f_{n+1};$$

as it appears also from (11), $v_n^{(1)} = y_n$ and $v_n^{(2)} = f_n$; replacing $u_n^{(3)}$ by a_n , one gets for the first Nordsieck formula

$$\left. \begin{aligned} y_{n+1} &= y_n + \frac{7h}{12} f_n + \frac{h}{6} a_n + \frac{5h}{12} f_{n+1}, \\ a_{n+1} &= -\frac{1}{2} f_n + \frac{1}{2} f_{n+1}. \end{aligned} \right\} \quad (12)$$

In the same way, we derive the modified Nordsieck formula for $k = 1$; by writing the Adams-Moulton formula in the form

$$y_{n+1} = y_n + \frac{8h}{12} f_n - \frac{h}{12} f_{n-1} + 0f_{n-2} + \frac{5h}{12} f_{n+1},$$

we get successively

$$\vec{p}^T = (1, \frac{8h}{12}, -\frac{h}{12}, 0), \quad \beta_{-1} = \frac{5h}{12};$$

$$\vec{u}_n^T = (y_n, f_n, f_{n-1}, f_{n-2}),$$

$$A = \begin{pmatrix} 1 & \frac{8h}{12} & -\frac{h}{12} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \vec{a} = \begin{pmatrix} \frac{5h}{12} \\ 1 \\ 0 \\ 0 \end{pmatrix};$$

the polynomial $P_n(x) = v_n^{(1)} + h(v_n^{(2)} \left(\frac{x - x_n}{h}\right) + v_n^{(3)} \left(\frac{x - x_n}{h}\right)^2 + v_n^{(4)} \left(\frac{x - x_n}{h}\right)^3)$ satisfies the conditions $P_n(x_n) = y_n$, $P'_n(x_{n-i}) = f_{n-i}$, $i = 0, 1, 2$; it follows that

$$\vec{u}_n = T \vec{v}_n \quad (13)$$

where

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 \\ 0 & 1 & -4 & 12 \end{pmatrix}; \quad T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{3}{4} & -1 & \frac{1}{4} \\ 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix};$$

$$\vec{q}^T = \vec{p}^T T = (1, \frac{7h}{12}, \frac{h}{16}, -\frac{h}{4})$$

$$B = T^{-1}AT = \begin{pmatrix} 1 & \frac{7h}{12} & \frac{h}{6} & -\frac{h}{4} \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{3}{4} & -\frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \end{pmatrix}; \quad \vec{b} = T^{-1}\vec{a} = \begin{pmatrix} \frac{5h}{12} \\ 1 \\ \frac{3}{4} \\ \frac{1}{6} \end{pmatrix};$$

(6) and (7) become explicitly

$$y_{n+1} = v_n^{(1)} + \frac{7h}{12} v_n^{(2)} + \frac{h}{6} v_n^{(3)} - \frac{h}{4} v_n^{(4)} + \frac{5h}{12} f_{n+1},$$

$$v_{n+1}^{(1)} = v_n^{(1)} + \frac{7h}{12} v_n^{(2)} + \frac{h}{6} v_n^{(3)} - \frac{h}{4} v_n^{(4)} + \frac{5h}{12} f_{n+1},$$

$$v_{n+1}^{(2)} = f_{n+1},$$

$$v_{n+1}^{(3)} = -\frac{3}{4} v_n^{(2)} - \frac{1}{2} v_n^{(3)} + \frac{3}{4} v_n^{(4)} + \frac{3}{4} f_{n+1},$$

$$v_{n+1}^{(4)} = -\frac{1}{6} v_n^{(2)} - \frac{1}{3} v_n^{(3)} + \frac{1}{2} v_n^{(4)} + \frac{1}{6} f_{n+1};$$

setting $v_n^{(3)} = a_n$ and $v_n^{(4)} = b_n$, we get for the first modified Nordsieck method:

$$y_{n+1} = y_n + \frac{7h}{12} f_n + \frac{h}{6} a_n - \frac{h}{4} b_n + \frac{5h}{12} f_{n+1},$$

$$a_{n+1} = -\frac{3}{4} f_n - \frac{1}{2} a_n + \frac{3}{4} b_n + \frac{3}{4} f_{n+1},$$

$$b_{n+1} = -\frac{1}{6} f_n - \frac{1}{3} a_n + \frac{1}{2} b_n + \frac{1}{6} f_{n+1}.$$

Another Way for Deriving the Nordsieck Formulas

A comparison of (12) with Reference 1 shows that Nordsieck gets its formula in a slightly different form. In fact, by working directly on the polynomial $P_n(x)$, one obtains the "true" Nordsieck formula much faster.

a) Nordsieck formula

Let $p < q$ be integers and $S(x:p,q)$ be the Lagrangian polynomial of degree $(q - p)$:

$$S(x:p,q) = \frac{(x - x_p)(x - x_{p+1}) \dots (x - x_{q-1})}{(x_q - x_p)(x_q - x_{p+1}) \dots (x_q - x_{q-1})};$$

the polynomial $P_n(x)$ of degree $k+1$ for the k^{th} Nordsieck formula is defined by the relations

$$P_n(x_n) = y_n, \quad (13)$$

$$P'_n(x_n) = f_{n-i}, \quad i = 0, 1, \dots, k; \quad (14)$$

$P_{n+1}(x)$ will be determined by

$$P_n(x_{n+1}) = y_{n+1},$$

$$P'_{n+1}(x_{n+1-i}) = f_{n+1-i}, \quad i = 0, 1, \dots, k; \quad (15)$$

$P'_n(x)$ and $P'_{n+1}(x)$, polynomial of degree k , are entirely determined by (14) and (15); one checks easily that

$$P'_{n+1}(x) = P'_n(x) + (f_{n+1} - P'_n(x_{n+1}))S(x:n+1-k, n+1); \quad (16)$$

let $f^P = P'_n(x_{n+1})$; by (10):

$$f^P = v_n^{(2)} + 2v_n^{(3)} + 3v_n^{(4)} + \dots + (k+1)v_n^{(k+2)};$$

equalling terms of some degree in (16) allows to compute $v_{n+1}^{(2)}, \dots, v_{n+1}^{(k+2)}$ as linear combination of $v_n^{(2)}, \dots, v_n^{(k+2)}$ and $(f_{n+1} - f^P)$.

It remains to compute $v_{n+1}^{(1)} = P_{n+1}(x_{n+1}) = y_{n+1}$; by Section 1, $y_{n+j} = Q(x_{n+1})$, where $Q(x)$ is the polynomial of degree $k+2$ satisfying the relations

$$Q(x_n) = y_n,$$

$$Q'(x_{n-i}) = f_{n-i}, \quad i = -1, 0, \dots, k;$$

one checks easily that

$$Q'(x) = P'_n(x) + (f_{n+1} - P'_n(x_{n+1}))S(x:n-k, n+1),$$

$$Q(x) = P_n(x) + (f_{n+1} - P'_n(x_{n+1})) \int_{x_n}^x S(y:n-k, n+1)dy;$$

consequently

$$y_{n+1} = v_{n+1}^{(1)} = P_n(x_{n+1}) + (f_{n+1} - f^P) \int_{x_n}^{x_{n+1}} S(y:n-k, n+1)dy \quad (17)$$

b) Modified Nordsieck formula

The polynomials $P_n(x)$ and $P_{n+1}(x)$ of degree $k+2$ for the k^{th} modified method are determined by the relations

$$P_n(x_n) = y_n, \quad P'_n(x_{n-i}) = f_{n-i}, \quad i = 0, 1, \dots, k+1;$$

$$P_{n+1}(x_{n+1}) = y_{n+1}, \quad P'_{n+1}(x_{n+1-i}) = f_{n+1-i}, \quad i = 0, 1, \dots, k+1;$$

then

$$P'_{n+1}(x) = P'_n(x) + (f_{n+1} - f^P)S(x:n - k, n + 1), \quad (18)$$

$$\text{with } f^P = P'_n(x_{n+1}) = v_n^{(2)} + 2v_n^{(3)} + 3v_n^{(4)} + \dots + (k+2)v_n^{(k+3)};$$

equalling the coefficients of some degree in (18) gives

$$v_{n+1}^{(2)}, \dots, v_{n+1}^{(k+3)} \text{ as linear combination of } v_n^{(2)}, \dots, v_n^{(k+3)} \text{ and } (f_{n+1} - f^P).$$

By Section 1, $y_{n+1} = Q(x_{n+1})$, where $Q(x)$ is the polynomial of degree $k+2$ which satisfies the relations:

$$Q(x_n) = y_n, \quad Q'(x_{n-i}) = f_{n-i}, \quad i = -1, 0, \dots, k;$$

then

$$Q'(x) = P'_n(x) + (f_{n+1} - P'_n(x_{n+1}))S(x:n - k, n + 1);$$

a comparison with (18) shows that $Q(x) = P_{n+1}(x)$, since

$$Q(x_{n+1}) = y_{n+1} = P_{n+1}(x_{n+1}):$$

$$P_{n+1}(x) = Q(x) = P_n(x) + (f_{n+1} - P'_n(x_{n+1})) \int_{x_n}^x S(x:n - k, n + 1) dx \quad (19)$$

$$y_{n+1} = P_n(x_{n+1}) + (f_{n+1} - f^P) \int_{x_n}^{x_{n+1}} S(x:n - k, n + 1) dx. \quad (20)$$

Remark

A comparison of (16) with (18) and (17) with (19) shows that the k^{th} modified Nordsieck formula is identical to the $(k+1)^{\text{th}}$ Nordsieck formula, except for the coefficient of $(f_{n+1} - f^p)$ in the expression of y_{n+1} which is the same in both k^{th} formulas.

Example. We consider again the case $k = 1$.

Nordsieck formula

$$P_n(x) = v_n^{(1)} + h(v_n^{(2)} \left(\frac{x - x_n}{h} \right) + v_n^{(3)} \left(\frac{x - x_n}{h} \right)^2);$$

by (13) and (14), $v_n^{(1)} = y_n$, $v_n^{(2)} = f_n$; we set $a_n = v_n^{(3)}$; by (16):

$$P'_{n+1}(x) = f_{n+1} + 2a_{n+1} \frac{x - x_{n+1}}{h} = f_n + 2a_n \frac{x - x_n}{h} + (f_{n+1} - f^p) \frac{x - x_n}{h};$$

it follows that $a_{n+1} = a_n + \frac{1}{2}(f_{n+1} - f^p)$; by (17):

$$y_{n+1} = P_n(x_{n+1}) + (f_{n+1} - f^p) \int_{x_n}^{x_{n+1}} \frac{(y - x_{n-1})(y - x_n)}{2h^2} dy,$$

$$y_{n+1} = y_n + h(f_n + a_n) + \frac{5h}{12} (f_{n+1} - f^p);$$

we get for the first Nordsieck formula the expressions:

$$y_{n+1} = y_n + h(f_n + a_n + \frac{5}{12} (f_{n+1} - f^p)),$$

$$f^p = f_n + 2a_n,$$

$$a_{n+1} = a_n + \frac{1}{2} (f_{n+1} - f^p).$$

Modified Nordsieck formula

As before, we substitute: $v_n^{(1)} = y_n$, $v_n^{(2)} = f_n$, $v_n^{(3)} = a_n$,
 $v_n^{(4)} = b_n$. By (19):

$$\begin{aligned} P_{n+1}(x) &= y_{n+1} + h(f_{n+1} \left(\frac{x-x_{n+1}}{h}\right) + a_{n+1} \left(\frac{x-x_{n+1}}{h}\right) + b_{n+1} \left(\frac{x-x_{n+1}}{h}\right)^3), \\ &= y_n + h(f_n \left(\frac{x-x_n}{h}\right) + a_n \left(\frac{x-x_n}{h}\right)^2 + b_n \left(\frac{x-x_n}{h}\right)^3 + (f_{n+1} - f^p) \int_{x_n}^x \frac{(y-x_{n-1})(y-x_n)}{2h^2} dy. \end{aligned}$$

by taking the successive derivative at x_{n+1} for the two members,
one gets

$$y_{n+1} = y_n + h(f_n + a_n + b_n + \frac{5}{12} (f_{n+1} - f^p)),$$

$$a_{n+1} = 2a_n + 3b_n + \frac{3}{4} (f_{n+1} - f^p),$$

$$b_{n+1} = b_n + \frac{1}{6} (f_{n+1} - f^p),$$

where

$$f^p = f_n + 2a_n + 3b_n.$$

SECTION 5. PREDICTOR FORMULAS

a) Nordsieck formula

For the k^{th} Nordsieck formula, the predictor value for x_{n+1} is given by $P_n(x_{n+1})$, where $P_n(x)$ is the polynomial of degree $k+1$ determined by the relations

$$P_n(x_n) = y_n, \quad P_n'(x_{n-i}) = f_{n-i}, \quad i = 0, 1, \dots, k;$$

a comparison with Section 1 shows that $P_n(x_{n+1})$ is nothing else then the value of the k^{th} Adams-Bashforth formula with a truncation error $D_k h^{k+2} y^{(k+2)}(\xi)$.

b) Modified Nordsieck formula

For the k^{th} modified Nordsieck formula, the predictor value for x_{n+1} is given by $P_n(x_{n+1})$, where $P_n(x)$ is the polynomial of degree $k+2$ determined by the relations

$$P_n(x_n) = y_n, \quad P_n'(x_{n-i}) = f_{n-i}, \quad i = 0, 1, \dots, k+1;$$

it follows that $P_n(x_{n+1})$ is the value of the $(k+1)^{\text{th}}$ Adams-Bashforth formula with a truncation error $D_{k+1} h^{k+3} y^{(k+3)}(\xi)$.

Remark

The modified formula has the advantage that the predictor and the corrector formulas have a truncation error of the same order.

SECTION 6. MODIFICATION OF THE LENGTH OF THE STEP

Suppose we integrate the differential equation $y' = f(x, y)$ from a to c with the step h for $x \leq b$ and h^* for $x > b$.

Let $u(x)$ be the discrete approximative solution for $x \leq b$ and $u^*(x)$ be the approximative solution for $x \geq b$. Let $z(x)$ be the exact solution of the differential equation such that $z(b) = u(b)$; as h varies, $z(x)$ varies also.

We use the notations

$$x_n = b + nh, \quad x_n^* = b + nh^*;$$

$$u_n = u(x_n), \quad u_n^* = u^*(x_n^*);$$

$$f_n = f(x_n, u_n) \quad \text{for } n \leq 0;$$

$$f_n^* = f(x_n^*, u_n^*) \quad \text{for } n \geq 0;$$

$$z_n = z(x_n), \quad z_n^* = z(x_n^*).$$

We suppose

- 1) $\rho = \frac{h}{h^*}$ is a constant as $h \rightarrow 0$;
- 2) all the hypotheses necessary to apply the self-stabilizing property and the generalized theorem in Section 2 are filled.

Because of the self-stabilizing property, it is suitable to compare $u^*(x)$ with $z(x)$. Let $d^*(x) = u^*(x) - z(x)$ for $x \geq b$. Replacing $d(x)$ by $d^*(x)$, h by h^* , q by 0, the generalized theorem gives the asymptotical relation

$$d^*(x) = h^{*p}(e_{\text{I}}(x) + O(h)) + h^{*s}(e_{\text{II}}(x) + O(h)) + h^{r+1}(e_{\text{III}}(x) + O(h)), \quad x \geq b;$$

the term $e_{\text{I}}(x)$ corresponds to the accumulation of the local truncation errors and is consequently independent of the starting procedure for the new h^* ; the

term $e_{II}(x)$ corresponds to the error at $x = b$; by hypothesis $d^*(b) = 0$, i.e., $s = -\infty$ and the second term of the asymptotical relation vanishes; only the term $e_{III}(x)$ depends on the starting procedure.

Nordsieck Method

Consider the k^{th} Nordsieck formula; according to Section 4, $P'_0(x)$ is the polynomial of degree k determined by the relations

$$P'_0(x_{-i}) = f_{-i}, \quad i = 0, 1, \dots, k;$$

the starting values f_{-i}^* for the new h^* are given by

$$f_{-i}^* = P'_0(x_{-i}^*), \quad i = 0, 1, \dots, k;$$

because of well-known properties of the interpolation there exists the constant coefficients (independent of h^*) a_{ij} such that

$$f_{-i}^* = \sum_{j=0}^k a_{ij} f_{-j}, \quad i = 0, 1, \dots, k, \quad (1)$$

$$z_{-i}'^* = \sum_{j=1}^k a_{ij} z_{-j}' + R_i, \quad i = 0, 1, \dots, k, \quad (2)$$

where the interpolation error R_i is given by

$$R_i = \frac{(x_{-i}^* - x_0)(x_{-i}^* - x_{-1}) \dots (x_{-i}^* - x_{-k})}{(k+1)!} z^{(k+2)}(\xi_i), \quad i = 0, 1, \dots, k$$

with $\min(x_{-i}^*, x_{-k}) \leq \xi_i \leq x_0$. Since $h = \rho h^*$ and $z^{(k+2)}(\xi_i) = z^{(k+2)}(b) + O(h)$, we can write

$$R_i = h^{*k+1} \frac{(-i)(\rho - i)(2\rho - i) \dots (k\rho - i)}{(k+1)!} z^{(k+2)}(b) + O(h^{*k+2}), \quad i = 0, 1, \dots, k;$$

subtracting (2) from (1), one gets

$$f_{-i}^* - z_{-i}'^* = -R_i + O(h^{*k+2}),$$

since by the self-stabilizing property $f_{-j} - z_{-j}' = O(h^{*k+3})$.

Using the notation

$$E_i = \frac{(-i)(\rho - i)(2\rho - i) \dots (k\rho - i)}{(k+1)!}$$

we get by the generalized asymptotical theorem

$$d^*(x) = h^{*k+2}(e_I(x) + O(h)) + h^{*k+2}(e_{III}(x) + O(h)), \quad b \leq x \leq c \quad (3)$$

where

$$\frac{d}{dx} e_I(x) = G(x, z(x))e_I(x) - C_k z^{(k+3)}(x), \quad e_I(b) = 0,$$

$$\frac{d}{dx} e_{III}(x) = G(x, z(x))e_{III}(x),$$

$$e_{III}(b) = - \{(\beta_1 + \beta_2 + \dots + \beta_k)E_1 + (\beta_2 + \beta_3 + \dots + \beta_k)E_2 + \dots + \beta_k E_k\} z^{(k+2)}(b);$$

we recall that $\beta_{-1}, \beta_0, \beta_1, \dots, \beta_k$ are the coefficients of the k^{th} Adams-Moulton method.

Remarks

- 1) The error due to the modification of the step is non-negligible in comparison with the error due to the accumulation of the truncation errors.
- 2) In (3), $G(x, z(x))$ is a scalar; for systems of differential equations, it suffices to replace $d^*(x)$, $e_I(x)$, $e_{III}(x)$, $z(x)$ by the corresponding vectors and $G(x, z(x))$ by a matrix.

Modified Nordsieck Formula

Consider the k^{th} modified formula; according to Section 4, $P'_0(x)$ is a polynomial of degree $k+1$ determined by the relations

$$P'_0(x_{-i}) = f_{-i}, \quad i = 0, 1, \dots, k+1;$$

the initial values f_{-i}^* for the new step h^* satisfy the relations

$$f_{-i}^* = P'_0(x_{-i}^*), \quad i = 0, 1, \dots, k;$$

there exist the constant coefficients b_{ij} (independent of h^*) such that

$$f_{-i}^* = \sum_{j=0}^{k+1} b_{ij} f_{-j}, \quad i = 0, 1, \dots, k, \quad (4)$$

$$z_{-i}^{\prime*} = \sum_{j=0}^{k+1} b_{ij} z_{-j}' + R_i, \quad i = 0, 1, \dots, k, \quad (5)$$

where R_i is the interpolation error

$$R_i = \frac{(x_{-i}^* - x_0)(x_{-i}^* - x_1) \dots (x_{-i}^* - x_{-k-1})}{(k+2)!} z^{(k+3)}(\xi_i), \quad i = 0, 1, \dots, k,$$

with $\min(x_{-i}^*, x_{-k-1}) \leq \xi_i \leq x_0$. We can write

$$R_i = h^{*k+2} H_i z^{(k+3)}(b) + O(h^{*k+3}),$$

$$H_i = \frac{(-i)(\rho - i) \dots ((k+1)\rho - i)}{(k+2)!};$$

because of the self-stabilizing property, if we subtract (5) from (4), one gets

$$f_{-i}^* - z_{-i}^{\prime*} = -R_i + O(h^{*h+3});$$

by the generalized asymptotical theorem, one obtains the relation

$$d^*(x) = h^{*k+2}(e_I(x) + O(h)) + h^{*k+3}(e_{III}(x) + O(h)) \quad (6)$$

where

$$\frac{d}{dx} e_I(x) = G(x, z(x)) - c_k z^{(k+3)}(x), \quad e_I(b) = 0;$$

$$\frac{d}{dx} e_{III}(x) = G(x, z(x))e_{III}(x),$$

$$e_{III}(b) = - \{(\beta_1 + \beta_2 + \dots + \beta_k)H_1 + (\beta_2 + \dots + \beta_k)H_2 + \dots + \beta_k H_k\} \vec{z}^{(k+3)}(b);$$

we recall that $\beta_{-1}, \beta_0, \beta_1, \dots, \beta_k$ are the coefficients of the k^{th} Adams-Moulton formula.

Remarks

- 1) Except for x closed to b , the error due to the modification of the step is negligible in (6);
- 2) the same relation (6) holds for systems of differential equations if one replaces the scalars by the convenient vectors or matrix.

SECTION 7. STARTING PROCEDURES

Although the starting procedure is not the most characteristic feature of the Nordsieck method, it is most painful to analyze. In order to make the job easier, we shall introduce a so-called "simplified starting procedure."

Given a differential equation

$$y' = f(x, y) \quad (1)$$

and initial value

$$y(b) = y_0,$$

the problem is to find a "good" initial \vec{v}_0 (or equivalently an initial polynomial $P_0(x)$).

We introduce the notations:

$$x_n = b + nh, \quad f_n = f(x_n, y_n), \quad n \geq 0;$$

the norm $||\vec{x}||$ of any vector \vec{x} is the sum of the absolute values of its components.

We suppose:

- 1) $f(x, y)$ satisfies the Lipschitz condition $|f(x, y^*) - f(x, y)| \leq \Omega |y^* - y|$,
 $b \leq x \leq c$;
- 2) the solution $z(x)$ of (1) with $z(b) = y_0$, possesses $(k+4)$ continuous derivatives.

Starting Procedure for the k^{th} Nordsieck Method

Starting with any polynomial $P_{0,0}(x)$ such that $P_{0,0}(x_0) = y_0$ and $P'_{0,0}(x_0) = f_0$, k steps of integration are executed forwards; then h is replaced by $-h$ (by the standard procedure of modification of the step) and k steps are executed backwards; $-h$ is replaced by h ; let $Q_{0,0}(x)$ be the polynomial obtained at that stage; the new polynomial $P_{0,1}(x)$ is determined by the conditions

$$P_{0,1}(x_0) = y_0, \quad P'_{0,1}(x_0) = f_0, \quad P''_{0,1}(x) = Q''_{0,0}(x);$$

the same procedure is repeated with $P_{0,1}(x)$; one gets a sequence of polynomials $P_{0,0}(x)$, $Q_{0,0}(x)$, $P_{0,1}(x)$, $Q_{0,1}(x)$,

Simplified Starting Procedure for the k^{th} Nordsieck Method

Starting with any polynomial $P_{0,0}(x)$ such that $P_{0,0}(x_0) = y_0$ and $P'_{0,0}(x_0) = f_0$, k steps of integration are executed forwards; let $P_{k,0}(x)$ be the polynomial obtained at this stage; $P_{0,1}(x)$ is then defined by

$$P_{0,1}(x_0) = y_0, \quad P'_{0,1}(x) = P'_{k,0}(x);$$

the same procedure is repeated for the new initial polynomial $P_{0,1}(x)$; a sequence $P_{0,0}(x)$, $P_{k,0}(x)$, $P_{0,1}(x)$, $P_{k,1}(x)$, ... is defined.

For the Nordsieck formulas, we have the following theorem.

Theorem 1

For h sufficiently small, all the polynomials, $P_{0,i}(x)$ and $Q_{0,i}(x)$ for the starting procedure, $P_{0,i}(x)$ and $P_{k,i}(x)$ for the simplified starting procedure, converge as i goes to infinity to the same polynomial $U(x)$ of degree $k+1$ satisfying the relations

$$U(x_0) = y_0, \tag{2}$$

$$U'(x_i) = f(x_i, U(x_i)), \quad i = 0, 1, \dots, k. \tag{3}$$

Starting Procedure for the k^{th} Modified Nordsieck Formula

$P_{0,i}(x)$ and $Q_{0,i}(x)$ are defined in the same way as for the Nordsieck formula, except that $(k+1)$ steps of integrations are executed forwards and backwards at each iteration.

Simplified Starting Procedure for the k^{th} Modified Nordsieck Formula

It is identical to the simplified starting procedure for the k^{th} Nordsieck method except that $(k+1)$ steps of integration are executed at each iteration and define the sequence $P_{0,0}(x)$, $P_{k+1,0}(x)$, $P_{0,1}(x)$, $P_{k+1,1}(x)$,

For the modified Nordsieck formula, we have the following theorem.

Theorem 2

For h small enough, all the polynomials, $P_{0,i}(x)$ and $Q_{0,i}(x)$ for the starting procedure, $P_{0,i}(x)$ and $P_{k+1,i}(x)$ for the simplified starting procedure, converge as i goes to infinity to the same and unique polynomial $U(x)$ of degree $k+2$ satisfying the relations

$$U(x_0) = y_0,$$

$$U'(x_i) = f(x_i, U(x_i)), \quad i = 0, 1, \dots, k+1.$$

Proof of the Convergence of the Starting Procedures

We shall consider only the case of the simplified starting procedure for the k^{th} Nordsieck method; all the other cases can be treated analogously.

We first recall that \vec{v}_n is the vector, the components of which are the coefficients of the polynomial

$$P_n(x) = v_n^{(1)} + h(v_n^{(2)} \left(\frac{x - x_n}{h} \right) + \dots + v_n^{(k+2)} \left(\frac{x - x_n}{h} \right)^{k+1})$$

and that \vec{u}_n is the vector of the components $y_n, f_n, f_{n-1}, \dots, f_{n-k}$. Since $P_n(x_n) = y_n$, $P'_n(x_{n-i}) = f_{n-i}$, $i = 0, 1, \dots, k$, there exists a regular matrix T such that $\vec{u}_n = T\vec{v}_n$. Let $\vec{u}_{n,i}$ and $\vec{v}_{n,k}$ be the components of $P_{n,i}(x)$ ($n = 0, 1, \dots, k$); proving the convergence of $P_{0,i}(x)$, (or $P_{k,i}(x)$) for $i \rightarrow \infty$ is equivalent to the proof of the convergence of $\vec{v}_{0,i}$ or $\vec{u}_{0,i}$.

Let for any \vec{u}_0 , $\vec{\kappa}(\vec{u}_0)$ be the new initial vector obtained after one iteration of the simplified starting procedure. From Appendix 1, it follows

that the convergence of $\vec{u}_{0,i}$ towards a limit vector independent of the initial vector $\vec{u}_{0,0}$ will be insured if we can show that there exists a constant $K < 1$ such that for any two \vec{u}_0 and \vec{u}_0^* ($u_0^{(1)} = u_0^{(1)*} = y_0$, $u_0^{(2)} = u_0^{(2)*} = f_0$), we have

$$||\vec{\kappa}(\vec{u}_0^*) - \vec{\kappa}(\vec{u}_0)|| \leq K ||\vec{u}_0^* - \vec{u}_0||.$$

We apply to \vec{u}_0 and \vec{u}_0^* the simplified starting procedure; the quantities concerning \vec{u}_0^* will be denoted by a star. Let

$$y_{n+1} = y_n + h(\beta_{-1}f_{n+1} + \dots + \beta_k f_{n-k})$$

be the Adams-Moulton formula; for y_1 and y_1^* , we get

$$y_1 = y_0 + h(\beta_{-1}f_1 + \beta_0 f_0 + \dots + \beta_k f_{-k}),$$

$$y_1^* = y_0^* + h(\beta_{-1}f_1^* + \beta_0 f_0^* + \dots + \beta_k f_{-k}^*);$$

but $|f_1^* - f_1| \leq \Omega |y_1^* - y_1|$; if $1 - |\beta_{-1}|h\Omega > 0$, then

$$|y_1^* - y_1| \leq \frac{1}{1 - |\beta_{-1}|h\Omega} (|y_0^* - y_0| + h(|\beta_0| |f_0^* - f_0| + \dots + |\beta_k| |f_{-k}^* - f_{-k}|));$$

from now on, we suppose that $h \leq \bar{h} = \frac{1}{2|\beta_{-1}|\Omega}$, let $B = |\beta_0| + |\beta_1| + \dots + |\beta_k|$; since $y_0^* = y_0$, we have

$$|y_1^* - y_1| \leq 2hB ||\vec{u}_0^* - u_0|| \quad \text{and} \quad |f_1^* - f_1| \leq 2h\Omega B ||\vec{u}_0^* - \vec{u}_0||; \quad (4)$$

in the same way, we get for the second step:

$$|y_2^* - y_2| \leq 2\{|y_1^* - y_1| + h(|\beta_0| |f_1^* - f_1| + |\beta_1| |f_0^* - f_0| + \dots + |\beta_k| |f_{1-k}^* - f_{1-k}|)\}$$

by (4):

$$|y_2^* - y_2| \leq 2\{2hB ||\vec{u}_0^* - u_0|| + h(2h\Omega B ||\vec{u}_0^* - \vec{u}_0|| + B ||\vec{u}_0^* - \vec{u}_0||)\};$$

$$|y_2^* - y_2| \leq Bh(3 + 2h\Omega) \|\vec{u}_0^* - \vec{u}_0\|;$$

since B and Ω are constant, we can write

$$|y_2^* - y_2| \leq o(h) \|\vec{u}_0^* - \vec{u}_0\|;$$

repeating the same process k times, we finally get

$$|y_i^* - y_i| = o(h) \|\vec{u}_0^* - \vec{u}_0\|, \quad i = 0, 1, \dots, k, \quad (5)$$

$$|f_i^* - f_i| = o(h) \|\vec{u}_0^* - \vec{u}_0\|, \quad i = 0, 1, \dots, k. \quad (6)$$

Let $R(x)$ and $R^*(x)$ be respectively the polynomials of degree k such that

$$R(x_i) = f_i, \quad i = 0, 1, \dots, k,$$

$$R^*(x_i) = f_i^*, \quad i = 0, 1, \dots, k;$$

$p_1, \dots, p_{k+2}, p_1^*, \dots, p_{k+2}^*$, components respectively of $\vec{\kappa}(\vec{u}_0)$ and $\vec{\kappa}(\vec{u}_0^*)$ are given by

$$p_1 = p_1^* = y_0,$$

$$p_2 = R(x_0) = p_2^* = R^*(x_0) = f_0,$$

$$p_3 = R(x_{-1}); \quad p_3^* = R^*(x_{-3}),$$

$$p_{k+2} = R(x_{-k}); \quad p_{k+2}^* = R^*(x_{-k});$$

then, there exist the constant coefficients (independent of h) such that

$$p_i = \sum_{j=0}^k a_{ij} f_j,$$

$$p_i^* = \sum_{j=0}^k a_{ij} f_j^*;$$

by (6)

$$|p_i - p_i^*| = O(h) \|\vec{u}_0^* - \vec{u}_0\|, \quad i = 0, 1, \dots, k$$

$$\|\vec{\kappa}(\vec{u}_0^*) - \vec{\kappa}(\vec{u}_0)\| = O(h) \|\vec{u}_0^* - \vec{u}_0\|;$$

this is the desired result since for h small enough $O(h) < 1$.

Remark that we have proved so far only the convergence of the polynomials $P_{0,i}(x)$; but the same arguments apply also for $P_{k,i}(x)$.

Proof of Theorem 1

We have already proved the convergence of the simplified starting procedure and of the starting procedure.

Let us call $U(x)$ the limit polynomial for the simplified starting procedure; by construction $U(x)$ satisfies (2) and (3) for $i = 0$; it remains to prove (3) for $i = 1, 2, \dots, k$. Let y_1, y_2, \dots, y_k be the values obtained by integrating with the k^{th} Nordsieck formula and starting polynomial $U(x)$. Because $U(x)$ is the limit polynomial, it follows that

$$U'(x_i) = f(x_i, y_i), \quad i = 0, 1, \dots, k; \quad (7)$$

since the k^{th} Nordsieck formula is equivalent to the k^{th} Adams-Moulton formula, it follows that $y_1 = R(x_1)$, where $R(x)$ is the polynomial of the degree $k+2$ satisfying the relations

$$R(x_0) = y_0 = U(x_0),$$

$$R'(x_i) = U'(x_i), \quad 0, -1, \dots, -k,$$

$$R'(x_1) = f(x_1, y_1);$$

by (7), $R'(x_1) = U'(x_1)$, and this implies that $R(x) = U(x)$, $y_1 = U(x_1)$, which proves (3) for $i = 1$. For $i = 2, 3, \dots, k$ the proof follows the same pattern.

We now discuss briefly the part of theorem 1 which concerns the starting procedure. Let $P_0(x)$ and $Q_0(x)$ be the limit of the polynomial $P_{0,i}(x)$ and $Q_{0,i}(x)$. We know that $P_0(x)$ and $Q_0(x)$ are independent of $P_{0,0}(x)$; let us choose $P_{0,0}(x) = U(x)$, where $U(x)$ is the polynomial which satisfies the relations (2) and (3) (Its existence has been established above.); by arguments similar to those used previously, it follows easily that $Q_{0,0}(x) = U(x) = P_{0,1}(x)$ and consequently $P_0(x) = Q_0(x) = U(x)$.

In order to avoid fastidious repetitions, we renounce to give the details of the proof of theorem 2.

Starting Errors in the Nordsieck Method

From theorem 1, it follows that the limit initial polynomial $U(x)$ is the same for the starting and the modified starting procedure. Let $P_0(x)$, $P_1(x)$, ... be the polynomials corresponding to the steps $0, 1, 2, \dots$; we know that $U(x) = P_0(x) = P_k(x)$ (and, in fact, also $U(x) = P_1(x), P_2(x), \dots, P_{k-1}(x)$). Let $z(x)$ be the exact solution of the differential equation with $z(x_0) = y_0$; we want to apply the asymptotical theorem of Section 2 with $q = k$; we have to determine

$$U(x_k) - z(x_k), \quad (8)$$

$$U'(x_i) - z'(x_i), \quad i = 0, 1, \dots, k. \quad (9)$$

Let $Q(x)$ be the polynomial of degree $k+1$ (as $U(x)$) satisfying the relations

$$Q(x_0) = y_0, \quad (10)$$

$$Q'(x_i) = z'(x_i), \quad i = 0, 1, \dots, k; \quad (11)$$

let

$$R_i = z(x_i) - Q(x_i), \quad i = 0, 1, \dots, k; \quad (12)$$

we have the relations

$$U(x_0) = y_0,$$

$$U'(x_i) = f(x_i, U(x_i)), \quad i = 0, 1, \dots, k, \quad (13)$$

$$Q(x_0) = y_0,$$

$$Q'(x_i) = f(x_i, Q(x_i) + R_i), \quad i = 0, 1, \dots, k; \quad (14)$$

let

$$H(x) = U(x) - Q(x), \quad (15)$$

$$t_i = H'(x_i), \quad i = 0, 1, \dots, k; \quad (16)$$

since $H(x_0) = 0$, there exist the constants a_{ij} independent of h such that

$$H(x) = h \sum_{j=0}^k a_{ij} t_j, \quad i = 0, 1, \dots, k; \quad (17)$$

we define the quantities g_i , $i = 0, 1, \dots, k$ by the relations

$$f(x_i, U(x_i)) - f(x_i, Q(x_i) + R_i) = g_i (U(x_i) - Q(x_i) - R_i), \quad i = 0, 1, \dots, k; \quad (18)$$

if $U(x_i) - Q(x_i) - R_i = 0$, we set $g_i = 0$.

We know that $|g_i| \leq \Omega$; subtracting (14) from (13), we get by (15), (16), (17), (18),

$$t_i = g_i h \sum_{j=0}^k a_{ij} t_j - g_i R_i, \quad i = 0, 1, \dots, k; \quad (19)$$

this is a linear system for the t_i 's if we consider R_i as known; we can regard t_i as function of h ; suppose that there exists a constant m such that $R_i = O(h^m)$, $i = 0, 1, \dots, k$; it follows the relation

$$t_i = -g_i R_i + O(h^{m+1}). \quad (20)$$

We only sketch the proof of (20). Since g_i is bounded, a_{ij} independent of h , (20) follows from (19), if we show that $t_i = O(h^m)$, $i = 0, 1, \dots, m$. Let \vec{t} be the vector of components t_i , \vec{w} the vector of components $-g_i R_i$ and B the matrix of elements $g_i a_{ij}$; (19) becomes

$$(I - hB)\vec{t} = \vec{w}; \quad (21)$$

we introduce the following notations: for any matrix C , the norm $||C||$ of C is the sum of the absolute value of its elements; norms of vectors and matrices have the properties:

- a) $||C\vec{x}|| \leq ||C|| \cdot ||\vec{x}||;$
- b) $||C + D|| \leq ||C|| + ||D||;$
- c) $||CD|| \leq ||C|| \cdot ||D||;$

by the Gerschgorin theorem, if $h \leq \frac{1}{2||B||}$, all the eigenvalues of hB lie in the circle of radius 0.5 around the origin; the matrix $I - hB$ is regular and its inverse possesses the convergent development

$$(I - hB)^{-1} = \sum_{n=0}^{\infty} h^n B^n;$$

by b) and c):

$$||(I - hB)^{-1}|| \leq \sum_{n=0}^{\infty} h^n ||B||^n = \frac{1}{1 - h||B||};$$

by (21) and a):

$$||\vec{t}|| \leq \frac{1}{1 - h||B||} ||\vec{w}|| \leq 2||\vec{w}|| = O(h^m), \text{ q.e.d.}$$

Now we can determine the errors (8) and (9); by (12) and (15), we have

$$U(x_i) - z(x_i) = H(x_i) - R_i, \quad i = 0, 1, \dots, k;$$

by (17) and (20):

$$U(x_i) - z(x_i) = -h \sum_{j=0}^k a_{ij} g_j R_j - R_i + O(h^{m+2}), \quad i = 0, 1, \dots, k; \quad (22)$$

by (11), (16) and (20):

$$U'(x_i) - z'(x_i) = t_i = -g_i R_i + O(h^{m+1}), \quad i = 0, 1, \dots, k. \quad (23)$$

In order to get more useful expressions, let us suppose (as in the asymptotical theorem) that $G(x, y) = \frac{\partial f}{\partial y}$ exists and satisfies the following requirements:

$$1) \quad |G(x, y^*) - G(x, y)| \leq \psi |y^* - y|, \quad \psi \text{ constant, } b \leq x \leq c;$$

2) $G(x, z(x))$ possesses a continuous derivative;

it follows that $g_i = G(x_i, y^*)$, where $y^* \in [U(x_i), z(x_i)]$; by (22), $U(x_i) - z(x_i) = O(h^m)$; because of the above two requirements, we have

$$g_i = G(x_i, z(x_i)) + O(h^m) = G(x_0, z(x_0)) + O(h) + O(h^m).$$

Replacing in (22) and (23), we get the final formulas

$$\left. \begin{aligned} U(x_i) - z(x_i) &= -hG(x_0, z(x_0)) \sum_{j=0}^k a_{ij} R_j - R_i + O(h^{m+2}), \\ U'(x_i) - z'(x_i) &= -G(x_0, z(x_0)) R_i + O(h^{m+1}); \end{aligned} \right\} \quad i = 0, 1, \dots, k \quad (24)$$

We recall that

$$G(x, y) = \frac{\partial f(x, y)}{\partial y};$$

$Q(x)$ is the polynomial of degree $k+1$ which satisfies the relations

$$Q(x_0) = z_0, \quad Q'(x_i) = z'(x_i), \quad i = 0, 1, \dots, k;$$

$$R_i = z(x_i) - Q(x_i);$$

a_{ij} are the coefficients of the relations

$$K(x_i) = K(x_0) + h \sum_{j=0}^k a_{ij} K'(x_i), \quad i = 0, 1, \dots, k, \quad (25)$$

where $K(x)$ is any polynomial of degree $k+1$.

Remark

We have considered the errors of the starting procedure for the limit polynomial $U(x)$; but practically, we must deal with the polynomial $P_{0,j}$ for some j ; we write

$$P_{0,j}(x_i) - z(x_i) = (P_{0,j}(x_i) - U(x_i)) + (U(x_i) - z(x_i)), \quad i = 0, 1, \dots, k;$$

by choosing $P_{0,0}(x) = y_0 + f_0(x - x_0)$, one gets easily from the proof of the convergence that

$$P_{0,j}(x_i) - U(x_i) = O(h^{2+j}), \quad i = 0, 1, \dots, k. \quad (26)$$

This result is valid for both the starting procedure and the modified starting procedure; for the simplified starting procedure, there is no reason for thinking that the order of the error in (26) may be greater than h^{2+j} ; however, it is very likely that for the starting procedure we have

$$P_{0,j}(x_i) - U(x_i) = O(h^{2+2j}), \quad i = 0, 1, \dots, k;$$

no proof of this statement has been established; if that is true, it means that the speed of convergence in comparison with the number of steps of integration to execute is the same for the starting and the modified starting procedure; the simplified starting procedure has the advantage of using only forward integration which remains closer to the true solution than the backward integration.

Example

We consider the starting procedure for the first Nordsieck method and suppose:

- 1) enough iterations have been executed for using (24), when $U(x)$ is replaced by $P_{0,j}(x)$,
- 2) all the assumptions necessary to the application of the asymptotic theorem are filled. A little arithmetic leads to the relations

$$a_{00} = a_{01} = 0; \quad a_{10} = a_{11} = \frac{1}{2};$$

$$R_0 = 0; \quad R_1 = -\frac{h^3}{12} z'''(x_0) + O(h^4);$$

$$P_{0,j}(x_1) - z(x_1) = \frac{h^3}{12} z'''(x_0) + O(h^4);$$

$$P'_{0,j}(x_0) = z(x_0) = 0;$$

if $u(x)$ denotes the approximative solution obtained with this starting procedure and the first Nordsieck formula and $d(x) = u(x) - z(x)$, we have by the asymptotical theorem (with $q = 1$)

$$d(x) = h^3(e_I(x) + O(h)) + h^3(e_{II}(x) + O(h)),$$

where

$$\frac{d}{dx} e_I(x) = G(x, z(x))e_I(x) + \frac{1}{24} z^{(4)}(x), \quad e_I(b) = 0;$$

$$\frac{d}{dx} e_{II}(x) = G(x, z(x))e_{II}(x), \quad e_{II}(b) = \frac{1}{12} z^{(3)}(b);$$

the term $e_{II}(x)$ represent the initial error due to the starting procedure.

Errors for the Starting Procedures for the Modified Nordsieck Formula

All we have said above can be repeated word for word for the k^{th} modified Nordsieck formula except that we must replace everywhere k by $k+1$.

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Example

For the first modified Nordsieck formula, we have:

$$a_{00} = a_{01} = a_{02} = 0;$$

$$a_{10} = \frac{5}{12}, \quad a_{11} = \frac{8}{12}, \quad a_{13} = \frac{-1}{12};$$

$$a_{20} = \frac{1}{3}, \quad a_{21} = \frac{4}{3}, \quad a_{22} = \frac{1}{3};$$

$$R_0 = 0, \quad R_1 = \frac{h^4}{24} z^{(4)}(b) + O(h^5), \quad R_2 = -\frac{h^5}{90} z^{(5)}(b) + O(h^6);$$

$$P_{0,j}(x_2) - z(x_2) = -\frac{1}{18} h^5 G(b, z(b)) z^{(4)}(b) + \frac{h^5}{90} z^{(5)}(b) + O(h^6);$$

$$P'_{0,j}(x_0) - z'(x_0) = 0; \quad P'_{0,j}(x_1) - z'(x_1) = -G(b, z(b)) \frac{h^4}{24} z^{(4)}(b) + O(h^5);$$

if $u(x)$ denotes the approximative solution and $d(x) = u(x) - z(x)$:

$$d(x) = h^3(e_I(x) + O(h)) + h^5(e_{II}(x) + e_{III}(x) + O(h)),$$

where

$$\frac{d}{dx} e_I(x) = G(x, z(x))e_I(x) + \frac{1}{24} z^{(4)}(x), \quad e_I(h) = 0;$$

$$\frac{d}{dx} e_{II}(x) = G(x, z(x))e_{II}(x), \quad e_{II}(b) = -\frac{1}{18} G(b, z(b))z^{(4)}(b) + \frac{1}{90} z^{(5)}(b);$$

$$\frac{d}{dx} e_{III}(x) = G(x, z(x))e_{III}(x), \quad e_{III}(b) = \frac{1}{288} G(b, z(b))z^{(4)}(b);$$

the starting error, represented by the term $e_{II} + e_{III}$ is negligible in comparison with the term e_I .

SECTION 8. CONCLUSION

The Nordsieck method, the modified Nordsieck method and all Nordsieck-like formulas considered in Section 4 look very elegant.

Let us consider once more the matrix formulation of the multistep method (3), (4), (5) and its Nordsieck-like equivalent (6), (7), (8). The classical multistep method is such that A and \vec{a} in (4) are the simplest possible, i.e., each normal step involves the least possible arithmetics. If \vec{v}_n is the vector of the iterative process (7) and \vec{v}_n^* the vector obtained after the modification of h , the Nordsieck-like method is such that the matrix D which transforms \vec{v}_n in \vec{v}_n^* is diagonal; the corresponding matrix for the classical multistep method has not this property.

In view of the fact that keeping the step constant is the rule and modifying it is the exception, one may prefer the old and classical formulation of the multistep method.

SECTION 9. TABLE OF RESULTS

For the Nordsieck method and the modified Nordsieck method ($k = 1, 2, 3, 4$) we give successively:

- 1) The Nordsieck formulation of the method.
- 2) The corresponding multistep corrector formula:

$$y(x + h) = y(x) + h(\beta_{-1}f(x + h) + \dots + \beta_k f(x - kh));$$

the truncation error is denoted by

$$T = Ch^{p+1} z^{(p+1)},$$

meaning that if $z(x)$ is a function which possesses $(p+1)$ continuous derivatives, then

$$z(x + h) = z(x) + h(\beta_{-1}z'(x + h) + \dots + \beta_k z'(x - kh)) + T,$$

$$T = Ch^{p+1} z^{(p+1)}(\xi), \quad x - kh \leq \xi \leq x + h.$$

- 3) The corresponding multistep predictor formula with the truncation error.
- 4) The error involved in the modification of the length of the step. Suppose the differential equation $y' = f(x, y)$ is integrated by the method we are considering with the step h^* for $x \leq b$ and the step h for $x > b$. Let $u(x)$ be the approximative solution, $z(x)$ be the exact solution such that $u(b) = z(b)$ and $d(x) = u(x) - z(x)$ for $x \geq b$; the error satisfies the asymptotic relation

$$d(x) = h^p(s(x) + O(h)) + h^q(t(x) + O(h));$$

the term $s(x)$ represents the accumulation of the truncation errors; the term $t(x)$ is due to the modification of the step; in the definitions of $s(x)$ and $t(x)$ we shall use the notations

$$g(x) = \frac{\partial f(x, y)}{\partial y} / y = z(x); \quad (1)$$

$$\rho = \frac{h^*}{h};$$

- 5) The error involved in the starting procedure. The differential equation $y' = f(x, y)$ is integrated for $x \geq b$ with prescribed initial value at $x = b$; if $u(x)$ is the approximative solution, $z(x)$ the exact solution such that $u(b) = z(b)$, $d(x) = u(x) - z(x)$, then for $x \geq b$

$$d(x) = h^p(s(x) + O(h)) + h^r(w(x) + O(h));$$

the first term represents the accumulation of the truncation errors and the second the error due to the starting procedure. The notation (1) will be used.

Nordsieck Formula $k = 1$

$$y(x + h) = y(x) + h\{f(x) + a(x) + \frac{5}{12} (f(x + h) - f^p)\},$$

$$f^p = f(x) + 2a(x),$$

$$a(x + h) = a(x) + \frac{1}{2} (f(x + h) - f^p).$$

Corrector Formula

$$y(x + h) = \frac{h}{12} (5f(x + h) + 8f(x) - f(x - h));$$

$$T = \frac{-h^4}{24} z^{(4)}.$$

Predictor Formula

$$y(x+h) = y(x) + \frac{h}{2} (3f(x) - f(x-h));$$

$$T = \frac{5h^3}{12} z^{(3)}.$$

Modification of h

$$d(x) = h^3(s(x) + O(h)) + h^3(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{1}{24} z^{(4)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = \frac{1-\rho}{24} z^{(3)}(b).$$

Starting Procedure

$$d(x) = h^3(s(x) + O(h)) + h^3(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{1}{24} z^{(4)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = \frac{1}{12} z^{(3)}(b).$$

Modified Nordsieck Formula $k=1$

$$y(x+h) = y(x) + h\{f(x) + a(x) + b(x) + \frac{5}{12} (f(x+h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x),$$

$$a(x+h) = a(x) + 3b(x) + \frac{3}{4} (f(x+h) - f^p),$$

$$b(x+h) = b(x) + \frac{1}{6} (f(x+h) - f^p).$$

Corrector Formula

$$y(x + h) = \frac{h}{12} (5f(x + h) + 8f(x) - f(x - h));$$

$$T = -\frac{h^4}{24} z^{(4)}.$$

Predictor Formula

$$y(x + h) = \frac{h}{12} (23f(x) - 16f(x - h) + 5f(x - 2h));$$

$$T = \frac{3h^4}{8} z^{(4)}.$$

Modification of h

$$d(x) = h^3(s(x) + O(h)) + h^4(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{1}{24} z^{(4)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = \frac{-(2\rho^2 - 3\rho + 1)}{72} z^{(4)}(b).$$

Starting Procedure

$$d(x) = h^3(s(x) + O(h)) + h^5(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{1}{24} z^{(4)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = \frac{-5}{96} g(b)z^{(4)}(b) + \frac{1}{90} z^{(5)}(b).$$

Nordsieck Formula $k = 2$

$$y(x + h) = y(x) + h\{f(x) + a(x) + b(x) + \frac{3}{8} (f(x + h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x),$$

$$a(x + h) = a(x) + 3b(x) + \frac{3}{4} (f(x + h) - f^p),$$

$$b(x + h) = b(x) + \frac{1}{6} (f(x + h) - f^p).$$

Corrector Formula

$$y(x + h) = y(x) + \frac{h}{24} (9f(x + h) + 19f(x) - 5f(x - h) + f(x - 2h));$$

$$T = - \frac{19h^5}{720} z^{(5)}.$$

Predictor Formula

$$y(x + h) = y(x) + \frac{h}{12} (23f(x) - 16f(x - h) + 5f(x - 2h));$$

$$T = \frac{3h^4}{8} z^{(4)}.$$

Modification of h

$$d(x) = h^4(s(x) + O(h)) + h^4(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{19}{720} z^{(5)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = \frac{1 - \rho^2}{36} z^{(4)}(b).$$

Starting Procedure

$$d(x) = h^4(s(x) + O(h)) + h^5(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{19}{720} z^{(5)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = -\frac{7}{144} g(b)z^{(4)}(b) + \frac{1}{90} z^{(5)}(b).$$

Modified Nordsieck Formula $k = 2$

$$y(x+h) = y(x) + h\{f(x) + a(x) + b(x) + c(x) + \frac{3}{8}(f(x+h) - f^P)\},$$

$$f^P = f(x) + 2a(x) + 3b(x) + 4c(x),$$

$$a(x+h) = a(x) + 3b(x) + 6c(x) + \frac{11}{12}(f(x+h) - f^P),$$

$$b(x+h) = b(x) + 4c(x) + \frac{1}{3}(f(x+h) - f^P),$$

$$c(x+h) = c(x) + \frac{1}{24}(f(x+h) - f^P).$$

Corrector Formula

$$y(x+h) = y(x) + \frac{h}{24}(9f(x+h) + 19f(x) - 5f(x-h) + f(x-2h));$$

$$T = -\frac{19h^5}{720} z^{(5)}.$$

Predictor Formula

$$y(x+h) = y(x) + \frac{h}{24}(55f(x) - 59f(x-h) + 37f(x-2h) - 9f(x-3h));$$

$$T = \frac{251h^5}{720} z^{(5)}.$$

Modification of h

$$d(x) = h^4(s(x) + O(h)) + h^5(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{19}{720} z^{(5)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = -\frac{\rho^3 - 2\rho + 1}{72} z^{(5)}(b).$$

Starting Procedure

$$d(x) = h^4(s(x) + O(h)) + h^5(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{19}{720} z^{(5)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = \frac{3}{80} z^{(5)}(b).$$

Nordsieck Formula $k = 3$

$$y(x + h) = y(x) + h\{f(x) + a(x) + b(x) + c(x) + \frac{251}{720} (f(x + h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x) + 4c(x),$$

$$a(x + h) = a(x) + 3b(x) + 6c(x) + \frac{11}{12} (f(x + h) - f^p),$$

$$b(x + h) = b(x) + 4c(x) + \frac{1}{3} (f(x + h) - f^p),$$

$$c(x + h) = c(x) + \frac{1}{24} (f(x + h) - f^p).$$

Corrector Formula

$$y(x + h) = y(x) + \frac{h}{720}(251f(x + h) + 646f(x) - 264f(x - h) + 106f(x - 2h) - 19f(x - 3h));$$

$$T = -\frac{3h^6}{160} z^{(6)}.$$

Predictor Formula

$$y(x + h) = y(x) + \frac{h}{24} (55f(x) - 59f(x - h) + 37f(x - 2h) - 9f(x - 3h));$$

$$T = \frac{251h^5}{720} z^{(5)}.$$

Modification of h

$$d(x) = h^5(s(x) + O(h)) + h^5(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{3}{160} z^{(6)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = -\frac{10\rho^3 - \rho - 9}{480} z^{(5)}(b).$$

Starting Procedure

$$d(x) = h^5(s(x) + O(h)) + h^5(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{3}{160} z^{(6)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = \frac{3}{80} z^{(5)}(b).$$

Modified Nordsieck Formula $k = 3$

$$y(x + h) = y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) + \frac{251}{720} (f(x + h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x)$$

$$a(x + h) = a(x) + 3b(x) + 6c(x) + 10d(x) + \frac{25}{24} (f(x + h) - f^p),$$

$$b(x + h) = b(x) + 4c(x) + 10d(x) + \frac{35}{72} (f(x + h) - f^p),$$

$$c(x + h) = c(x) + 5d(x) + \frac{5}{48} (f(x + h) - f^p),$$

$$d(x + h) = d(x) + \frac{1}{120} (f(x + h) - f^p).$$

Corrector Formula

$$y(x + h) = y(x) + \frac{h}{720} (251f(x + h) + 646f(x) - 264f(x - h) + 106f(x - 2h) - 19f(x - 3h));$$

$$T = - \frac{3h^6}{160} z^{(6)}.$$

Predictor Formula

$$y(x + h) = y(x) + \frac{h}{720} (1901f(x) - 2774f(x - h) + 2616f(x - 2h) - 1274f(x - 3h)) \\ + 251f(x - 4h);$$

$$T = \frac{95h^6}{288} z^{(6)}.$$

Modification of h

$$d(x) = h^5(s(x) + O(h)) + h^6(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{3}{160} z^{(6)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = -\frac{48\rho^4 - 7\rho^2 - 180\rho + 67}{576} z^{(6)}(b).$$

Starting Procedure

$$d(x) = h^5(s(x) + o(h)) + h^7(w(x) + o(h));$$

$$s'(x) = g(x)s(x) + \frac{3}{160} z^{(6)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = -\frac{959}{17280} g(b)z^{(6)}(b) + \frac{8}{945} z^{(7)}(b).$$

Nordsieck Formula $k = 4$

$$y(x+h) = y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) + \frac{95}{288} (f(x+h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x),$$

$$a(x+h) = a(x) + 3b(x) + 6c(x) + 10d(x) + \frac{25}{24} (f(x+h) - f^p),$$

$$b(x+h) = b(x) + 4c(x) + 10d(x) + \frac{35}{72} (f(x+h) - f^p),$$

$$c(x+h) = c(x) + 5d(x) + \frac{5}{48} (f(x+h) - f^p),$$

$$d(x+h) = d(x) + \frac{1}{120} (f(x+h) - f^p).$$

Corrector Formula

$$y(x+h) = y(x) + \frac{h}{1440} \{475f(x+h) + 1427f(x) - 798f(x-h) + 482f(x-2h) \\ - 173f(x-3h) + 27f(x-4h)\},$$

$$T = -\frac{863h^7}{60480} z^{(7)}.$$

Predictor Formula

$$y(x+h) = y(x) + \frac{h}{720} (1901f(x) - 2774f(x-h) + 2616f(x-2h) \\ - 1274f(x-3h) + 251f(x-4h))$$

$$T = \frac{95h^6}{288} z^{(6)}.$$

Modification of h

$$d(x) = h^6(s(x) + O(h)) + h^6(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{863}{60480} z^{(7)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = -\frac{1}{14400} (240\rho^4 - 35\rho^2 - 1792\rho + 1587)z^{(6)}(b),$$

Starting Procedure

$$d(x) = h^6(s(x) + O(h)) + h^7(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{863}{60480} z^{(7)}(x), \quad s(b) = 0.$$

$$w'(x) = g(x)w(x), \quad w(b) = -\frac{653}{21600} g(b)z^{(6)}(b) + \frac{8}{945} z^{(7)}(b).$$

Modified Nordsieck Formula $k = 4$

$$y(x + h) = y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) + e(x) + \frac{95}{288} (f(x + h) - f^p)\},$$

$$f^p = f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x) + 6e(x),$$

$$a(x + h) = a(x) + 3b(x) + 6c(x) + 10d(x) + 15e(x) + \frac{137}{120} (f(x + h) - f^p),$$

$$b(x + h) = b(x) + 4c(x) + 10d(x) + 20e(x) + \frac{5}{8} (f(x + h) - f^p),$$

$$c(x + h) = c(x) + 5d(x) + 15e(x) + \frac{17}{96} (f(x + h) - f^p),$$

$$d(x + h) = d(x) + 6e(x) + \frac{1}{40} (f(x + h) - f^p),$$

$$e(x + h) = e(x) + \frac{1}{720} (f(x + h) - f^p).$$

Corrector Formula

$$y(x + h) = y(x) + \frac{h}{1440} (475f(x + h) + 1427f(x) - 798f(x - h) + 482f(x - 2h) \\ - 173f(x - 3h) + 27f(x - 4h))$$

$$T = - \frac{863h^7}{60480} z^{(7)}.$$

Predictor Formula

$$y(x + h) = y(x) + \frac{h}{1440} (4227f(x) - 7673f(x - h) + 9482f(x - 2h) \\ - 6798f(x - 3h) + 2627f(x - 4h) - 425f(x - 5h)).$$

$$T = \frac{19087h^7}{60480} z^{(7)}.$$

Modification of h

$$d(x) = h^6(s(x) + O(h)) + h^7(t(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{863}{60480} z^{(7)}(x), \quad s(b) = 0;$$

$$t'(x) = g(x)t(x), \quad t(b) = -\frac{1}{1152} (16\rho^5 - 3\rho^3 - 41\rho + 28)z^{(7)}(b).$$

Starting Procedure

$$d(x) = h^6(s(x) + O(h)) + h^7(w(x) + O(h));$$

$$s'(x) = g(x)s(x) + \frac{863h^7}{60480} z^{(7)}(x), \quad s(b) = 0;$$

$$w'(x) = g(x)w(x), \quad w(b) = \frac{275}{12096} z^{(7)}(b).$$

APPENDIX 1

Let \vec{x} be a vector of m components x_1, x_2, \dots, x_m , $\vec{f}(x)$ be a vector function of \vec{x} with components $f_1(\vec{x}), \dots, f_m(\vec{x})$. For any vector \vec{z} , $||\vec{z}|| = \sum_{i=1}^m |z_i|$ is the norm of \vec{z} .

Theorem

If for any two \vec{x} and \vec{x}^* , there exists a constant $c < 1$ such that

$$||\vec{f}(\vec{x}^*) - \vec{f}(x)|| \leq c ||\vec{x}^* - \vec{x}||, \quad (1)$$

then

a) the equation

$$\vec{x} = \vec{f}(\vec{x}) \quad (2)$$

possess one and only one solution \vec{y} ;

b) for any \vec{x}_0 , the sequence $\vec{x}_0, \vec{x}_1, \dots, \vec{x}_n, \dots$ defined by

$$\vec{x}_{n+1} = \vec{f}(\vec{x}_n) \quad (3)$$

converges towards \vec{y} .

Proof

This theorem is only a particular formulation of classical results in Banach spaces theory. The proof results from the two statements:

- a) the equation (2) has at most one solution;
 - b) the sequence defined by (3) converges towards a solution of (2).
- a) Let \vec{y} and \vec{y}^* be two solutions of (2). By (1):

$$||\vec{y}^* - \vec{y}|| = ||\vec{f}(\vec{y}^*) - \vec{f}(\vec{y})|| \leq c ||\vec{y}^* - \vec{y}||,$$

and consequently $||\vec{y}^* - \vec{y}|| = 0$.

- b) Let $x_{n,1}, x_{n,2}, \dots, x_{n,m}$ be the components of \vec{x}_n . In order to prove the convergence, it suffices to show that for every i , $i = 1, \dots, m$, the sequence $x_{n,i}$ is a Cauchy sequence. Let $p > q \geq N$:

$$|x_{p,i} - x_{q,i}| \leq ||\vec{x}_p - \vec{x}_q|| \leq ||\vec{x}_q - \vec{x}_{p-1}|| + ||\vec{x}_{p-1} - \vec{x}_{p-2}|| + \dots + ||\vec{x}_{q+1} - \vec{x}_q||; \quad (4)$$

by (1):

$$||\vec{x}_2 - \vec{x}_1|| = ||\vec{f}(\vec{x}_1) - \vec{f}(\vec{x}_0)|| \leq c ||\vec{x}_1 - \vec{x}_0||,$$

$$||\vec{x}_3 - \vec{x}_2|| = ||\vec{f}(\vec{x}_2) - \vec{f}(\vec{x}_1)|| \leq c ||\vec{x}_2 - \vec{x}_1|| \leq c^2 ||\vec{x}_1 - \vec{x}_0||,$$

$$||\vec{x}_{n+1} - \vec{x}_n|| \leq c^n ||\vec{x}_1 - \vec{x}_0||;$$

replacing in (4), we get

$$|x_{p,i} - x_{q,i}| \leq ||\vec{x}_1 - \vec{x}_0|| \sum_{n=q}^{p-1} c^n \leq ||\vec{x}_1 - \vec{x}_0|| \sum_{n=N}^{\infty} c^n = \frac{c^N}{1-c} ||\vec{x}_1 - \vec{x}_0||;$$

since $c < 1$ and $||\vec{x}_1 - \vec{x}_0||$ is constant, for given ϵ , it is possible to find N such that $||x_{p,i} - x_{q,i}|| \leq \epsilon$, $p, q \geq N$.

Let $\vec{y} = \lim_{n \rightarrow \infty} \vec{x}_n$; by (3):

$$y = \lim_{n \rightarrow \infty} \vec{x}_{n+1} = \lim_{n \rightarrow \infty} \vec{f}(\vec{x}_n);$$

by (1), $\vec{f}(\vec{x})$ is continuous and consequently $\lim_{n \rightarrow \infty} \vec{f}(\vec{x}_n) = \vec{y}$, q.e.d.

APPENDIX 2. SOME EQUIVALENT "METHODS"

Let $y' = f(x, y)$ be a differential equation to be integrated at $x_0, x_1, \dots, x_n, \dots, x_n = x_0 + nh$. We consider h as a fixed number and denote by y_n the approximative solution at x_n . We define two processes for computing y_n .

Method 1

For $n \geq 0$:

$$y_{n+1} = \vec{p}^T \vec{v}_n + \gamma f_{n+1}, \quad (1)$$

$$\vec{v}_{n+1} = A \vec{v}_n + f_{n+1} \vec{a}, \quad (2)$$

$$f_{n+1} = f(x_{n+1}, y_{n+1}); \quad (3)$$

A is a matrix of order m with constant elements; \vec{p} and \vec{a} are constant vectors; γ is a scalar constant; \vec{v}_0 is the initial vector. For given \vec{v}_0 , this method allows to compute successively y_1, y_2, \dots , provided that the non-linear system of equations (1), (3), has always a unique solution (we shall always suppose that it is the case).

Method 2

For $n \geq 0$:

$$y_{n+1} = s_{n+1} + t_{n+1}, \quad (4)$$

$$s_{n+1} = \alpha_0 s_n + \alpha_1 s_{n-1} + \dots + \alpha_j s_{n-j} + \beta_0 f_n + \beta_1 f_{n-1} + \dots + \beta_k f_{n-k} + \delta f_{n+1} \quad (5)$$

$t_{n+1} = 0$ for all n , except for $n = w, w+1, \dots, w+p$, where $w = \max(0, k-j)$;

$j, k, p, \alpha_0, \dots, \alpha_j, \beta_0, \dots, \beta_k, \delta$ are the constants which characterize the method 2. Suppose that $j \geq -1, k \geq -1, p \geq -1$; if $j = -1$, there are no α -terms in (5); if $k = -1$, there are no β -terms in (5); if $p = -1$, all t -terms are zero and the method (2) becomes the classical multistep method; furthermore $\alpha_j \neq 0$ if $j \geq 0, \beta_k \neq 0$ if $k \geq 0$; $t_{w+1}, t_{w+2}, \dots, t_{w+p+1}, s_0, s_{-1}, \dots, s_{-j}, f_0, f_{-1}, f_{-2}, \dots, f_{-k}$ are the initial values; let \vec{u}_0 denote the vector of dimension $p+j+k+3$, the components of which are the initial values.

Equivalence

A particular method 1 and a particular method 2 are said equivalent if there exist two rectangular matrices P and Q with constant coefficients (i.e., independent of $f(x,y), \vec{u}_0, \vec{v}_0$) such that if $\vec{v}_0 = P\vec{u}_0$ for any given \vec{u}_0 or $\vec{u}_0 = Q\vec{v}_0$ for any given \vec{v}_0 , then methods 1 and 2 give the same values for y_n , $n = 1, 2, \dots$.

Theorem 1

- a) For any method 1, there exists an equivalent method 2.
- b) For any method 2, there exists an equivalent method 1.
- c) Two methods of type 2 are equivalent if and only if all their characteristic parameters are identical.

Theorem 1 has the following practical meaning: all methods 1 are "essentially equivalent" to a classical multistep method. These methods 1, which are not rigorously equivalent to a classical multistep method (i.e., which are equivalent to a method 2 with $p \geq 0$), do not seem to have much interest. Therefore all "interesting methods 1" can be discussed in terms of multistep formulas.

The proof of theorem 1 which is rather painful and long, will not be given here.

We now consider two other methods, method III (multistep method) and method IV.

Method III

For $n \geq 0$:

$$y_{n+1} = \alpha_0 y_n + \alpha_1 y_{n-1} + \dots + \alpha_j y_{n-j} + \beta_0 f_n + \beta_1 f_{n-1} + \dots + \beta_{j+p} f_{n-j-p} + \gamma f_{n+1},$$

$$f_{n+1} = f(x_{n+1}, y_{n+1});$$

$j, p, \alpha_0, \dots, \alpha_j, \beta_0, \dots, \beta_{j+p}, \gamma$ are the constants which characterize the method; $j \geq 0, \alpha_j \neq 0; p \geq 0$; if $p > 0$, then $\beta_{j+p} \neq 0$; $y_0, y_{-1}, \dots, y_{n-j}, f_0, f_{-1}, \dots, f_{-j-p}$ are the initial values.

Method IV

For $n \geq 0$:

$$z_{n+1} = \alpha_0 z_{0,n} + \alpha_1 z_{1,n} + \dots + \alpha_j z_{j,n} - \delta_{j+1} g_n - \delta_{j+2} g_{n-1} - \dots - \delta_{j+p} g_{n-p+1} + \gamma g_{n+1},$$

$$g_{n+1} = f(x_{n+1}, y_{n+1})$$

$$z_{0,n+1} = \alpha_0 z_{0,n} + \alpha_1 z_{1,n} + \dots + \alpha_j z_{j,n} + (\delta_j + \gamma) g_{n+1},$$

$$z_{i,n+1} = z_{i-1,n} + \delta_{j-i} g_{n+1}, \quad i = 1, 2, \dots, j;$$

$j, p, \alpha_0, \dots, \alpha_j, \delta_0, \dots, \delta_j, \delta_{j+1}, \delta_{j+p}, \gamma$ are the constraints which characterize the method; $j \geq 0, \alpha_j \neq 0, p \geq 0$; if $p = 0$, there are no δ -terms in the expression of z_{n+1} ; if $p > 0, \delta_{j+p} \neq 0$; $z_{0,0}, \dots, z_{j,0}, g_0, g_{-1}, \dots, g_{1-p}$ are the initial values.

Theorem 2

If the coefficients $\beta_0, \dots, \beta_{j+p}$ of method III and the coefficients $\delta_0, \delta_1, \dots, \delta_{j+p}$ of method IV satisfy the relations

$$\beta_i = \sum_{r=0}^j \alpha_r \delta_{j+i-r} - \delta_{j+1+i}, \quad i = 0, 1, \dots, p+j; \quad (6)$$

where one defines $\delta_{j+p+i} \equiv 0$ for $i > 0$, and if the relations

$$g_{n-i} = f_{n-i}, \quad i = 0, 1, \dots, p-1, \quad (7)$$

$$z_{i,n} = y_{n-i} + \sum_{r=0}^{p+i} \delta_{j-i+r} f_{n-r}, \quad i = 0, 1, \dots, j, \quad (8)$$

are satisfied for $n = 0$, then they also hold for $n = 1, 2, \dots$ and $z_n = y_n$.

The proof of theorem 2 by induction with respect to n is straightforward and will not be given here.

Remarks

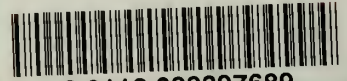
- 1) For given β_i , (6) is a linear system of equations for $\delta_0, \dots, \delta_{j+p}$; as it is easy to show the matrix of the system is triangular; all diagonal elements are equal to α_j ; by hypothesis $\alpha_j \neq 0$ and the matrix is regular. This establishes the equivalence of methods III and IV.
- 2) One iteration in methods III or IV involves practically the same number of operations. However method III needs $2j + p + 2$ "past values" $y_n, y_{n-1}, \dots, y_{n-j}, f_n, \dots, f_{n-j-p}$ (supposing $p > 0$), but method IV needs only $j + p + 1$ "past values," $z_{0,n}, z_{1,n}, \dots, g_n, \dots, g_{n+1-p}$. That curious property would have perhaps been useful in a time where the computers had only small memories. One can prove that for two different sets of initial values in method IV, the corresponding sequences of values z_n are also different; this is not the case for method III, since the equations (7), (8) considered as a linear system for $y_n, \dots, y_{n-j}, f_n, \dots, f_{n-j-p}$ have more unknowns than equations; however, the corresponding matrix has the maximal rank,

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